XX.—*A Memoir on Plane Analytic Geometry.*—*By* Asutosh Mukhopadhyay, M. A., F. R. A. S., F. R. S. E. *Communicated by* The Hon'ble Mahendralal Sircar, M. D., C. I. E.

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(With three Wood-cuts.)

Contents.

§. 1. *Introduction.*

§. 1. **Object and Scope.**—It is my object in the present paper to bring together a number of theorems in plane analytic geometry which have accumulated in my hands during my study of that subject. Some of the simpler of these theorems have already been given in my Lectures on Plane Analytic Geometry, now in course of delivery at the Indian Association for the Cultivation of Science; a few have been enunciated elsewhere without demonstration; most of the propositions, however, are here published for the first time. I believe that either the theorems themselves, or the methods of establishing them are original; and, except in a very few instances where I have inserted well-known results for the sake of avoiding disconnectedness, I have considered them either for the purpose of giving a proof simpler and more complete than that usually given, or with a view to throw light on the connection between the various parts of the subject. As the different sections of this paper are, to a great extent, practically independent of each other, for the sake of facility of reference, an outline of the principal topics discussed is added above.*

§. 2. *Basis of Analytical Geometry.*

§. 2. **Analysis and Geometry.**—The notion of either space or number, or of both, lies at the root of every department of mathematics. Analysis is the science of number; geometry is the science of space; but, as space is homogeneous, and, as every homogeneous substance can, by the choice of a unit, be represented by a number, space can be, for mathematical purposes, represented by numbers; hence, the *possibility* of applying analytical methods to geometrical investigations, and of founding a science of analytical geometry. This possibility was first *realized* into practice by the illustrious French mathematician René Descartes, who invented the method of coordinates. With respect

* For a full analysis of this paper, see the Proceedings for 1887, pp. 232-235.

to this method, there are two points which ought to be most carefully noticed. In the first place, to determine the position of any point, we must choose an origin, and, then, fix the position of the point by its coordinates, which may be defined to be independent quantities of the same order which fix the position of a point; we see, then, that the two essentially distinct ideas of origin and coordinates are fundamental in this theory; and, if we consider the matter for a moment, we find that the same two ideas are ever present in every system of coordinates that we may choose. Thus, looking to a comparatively modern part of the subject, the theory of Elliptic Coordinates, we see that the position of any point is determined by the lengths of the semi-axes of the conics which can be drawn through that point confocal to a given conic, called the primitive conic; here, then, the point-origin of the Cartesian system has been replaced by the fundamental conic, and the ordinate and abscissa have been replaced by the semi-axes of two conics. Hence, we conclude that in every system, we must have an origin, which is, as it were, a unit or symbol of reference, and which may be a point or a conic, or any other figure, according to the system we choose; and, having fixed our origin, we determine the position of a point by coordinates, which may be lines straight or curved, or any other geometrical figure; the only essential ideas being those of a symbol of reference, and of the independence of the quantities which fix the position of the point relatively to that origin or symbol of reference.

Having thus fixed the position of a point, we next consider how to represent a curve. A curve is defined to be an assemblage of points arranged according to a definite law; the equation of a curve, therefore, is the analytical representation of that geometrical relation which must subsist between the coordinates of a point, in order that that point may be on the given curve. In other words, the equation of a curve may be defined to be the analytical representation of some geometrical property of the curve; and, as a curve has an infinite number of geometrical properties, the question naturally suggests itself whether the analytical representation of each of these properties will give a different equation of the curve. As a matter of fact, we do know that, in whatever way we may derive the equation of a curve, we are led to equations which are apparently different from each other, but which are really not distinct, and which may all be made to coincide by suitable transformations. Indeed, if the reverse had been the case, it would have been manifestly impossible to create a science of analytic geometry; and the reason why all the equations of a curve are really identical is a simple outcome of the fact that all the innumerable geometrical properties of any curve are dependent on each other: the truth of any one being assumed, the others can be deduced from it as necessary mathematical consequences. We see, therefore, that though a curve has an infinite number of geometrical properties, it can have only one equation, and this accords with the great Law of Nature that, *in every natural system, there can be only one relation between the component parts.* This, then, is the second fact which made possible the very existence of Analytical Geometry.

From what has been pointed out above, it is evident that the equation of a curve is, as it were, a convenient repository of all theorems connected with it, and all its properties may be established by algebraic transformation of the equation. From this, as well as from the fundamental relation between analysis and geometry noted above, it is clear that, to every algebraic transformation, there corresponds a geometrical fact, and *vice versâ.* Take, for example, the subject of the transformation of coordinates. We all know that transformation is of two kinds; it may be a change to new axes, parallel to the old ones, through a new origin, which may conveniently be termed **Translation-transformation**; or, again, the transformation may be to new axes, inclined to the old ones, through the old origin, which may be called **Rotation-transformation**; if, in any case, both these kinds are combined, we may call it **Compound-transformation**; and from the known algebraical formulæ for compound transformation, it is clear that this geometrical process is nothing but the exact counterpart of the algebraic process of linear transformation. Similarly, it may be remarked that the problem of inversion is a case of quadric transformation.

§§. 3—5. *The Right Line.*

§. 3. **The Line at Infinity.**—The equation of any line being

$$\frac{x}{a}+\frac{y}{b}=1,$$

where a, b are the intercepts on the co-ordinate axes, the equation of the line which is at an infinite distance from the origin is obtained by substituting herein

$$a=b=\infty,$$

which gives

$$1=0.$$

Without any real change of generality, we may write this

$$\lambda=0$$

where λ is any constant; this, then, is the equation of the line at infinity; it will be of use in determining the asymptotes of the conic given by the general equation of the second degree (§. 12).

§. 4. **Coordinates of intersection of two lines.** The following method of investigating the condition that the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

may represent two right lines, is shorter than the proofs usually given, and has, besides, the advantage of furnishing at once the coordinates of the point of intersection of the lines represented by the equation.

Let (x', y') be the point of intersection of the lines; removing our origin to this point, the equation becomes

$$ax^2+2hxy+by^2+2g'x+2f'y+c'=0 \qquad \text{............ (1)}$$

where
$$g'=ax'+hy'+g,$$
$$f'=hx'+by'+f,$$
$$c'=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c.$$

But the equation (1) now represents a pair of lines through the origin, and, as such, it ought to be homogeneous in the second degree; therefore, each of the quantities g', f', c' must vanish separately, which gives

$$ax'+hy'+g=0 \qquad \text{............ (2)}$$
$$hx'+by'+f=0 \qquad \text{............ (3)}$$
$$ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0 \qquad \text{............ (4)}$$

Multiplying (2) by x', (3) by y', and subtracting the sum of the products from (4), we get

$$gx'+fy'+c=0 \qquad \text{............ (5)}$$

From (2) and (3), we have

$$x'=\frac{hf-bg}{ab-h^2},\quad y'=\frac{hg-af}{ab-h^2}, \qquad \text{............ (6)}$$

which are, accordingly, the coordinates of the point of intersection of the lines represented by the given equation. Eliminating x', y', from (2), (3), (5), we have the condition that the discriminant must vanish in order that the equation may represent two right lines, *viz.*,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0 \qquad \text{............ (7)}$$

As the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is transformed to

$$ax^2+2hxy+by^2=0$$

when the axes are removed to the point of intersection of the lines, it follows that, as the angle between the lines is not altered in magnitude by the transformation, the angle between the lines given by the general equation of the second degree is the same as that between the lines

$$ax^2+2hxy+by^2=0.$$

The quantity c', which occurs in this investigation, may be called the point-function of the conic.

Definition.—The **point-function** of any curve with respect to any point is the function which is obtained by substituting the coordinates of that point in the expression the vanishing of which gives the equation of the curve. It is clear that the point-function with respect to any point on the curve itself is zero, while the point-function with respect to the origin is the absolute term in the equation of the curve.

§. 5. **Area of a Triangle.**—If the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \text{............ (8)}$$

represents a pair of right lines, to investigate the area of the triangle formed by these two lines with the line

$$lx+my=n. \qquad \text{............ (9)}$$

Remove the origin to the point

$$\left(\frac{hf-bg}{ab-h^2}, \frac{hg-af}{ab-h^2}\right),$$

which is the point of intersection of the pair of lines represented by (8). The two equations then become

$$ax^2+2hxy+by^2=0 \qquad \text{............ (10)}$$

and

$$l\left(x+\frac{hf-bg}{ab-h^2}\right)+m\left(y+\frac{hg-af}{ab-h^2}\right)=n,$$

or

$$lx+my=p, \qquad \text{............ (11)}$$

where $$p=\frac{l(hf-bg)+m(hg-af)+n(h^2-ab)}{h^2-ab} \text{............ (12)}$$

Now, suppose that the lines in (10) are made up of the two

$$y-m_1x=0,\ y-m_2x=0, \qquad \text{............ (13), (14)}$$

so that

$$m_1+m_2=-\frac{2h}{b} \qquad \text{............ (15)}$$

$$m_1\ m_2=\ \frac{a}{b} \qquad \text{............ (16)}$$

whence $$m_1^2+m_2^2=\frac{4h^2-2ab}{b^2} \qquad \text{............ (17)}$$

The coordinates of the point of intersection of (11) with (13) are given by

$$x=\frac{p}{l+mm_1},\ y=\frac{m_1p}{l+mm_1}.$$

If, therefore, δ_1 is the length of the line intercepted between the new origin (which is the point of intersection of the pair of lines) and the point of intersection of (11) with (13), we have

$$\delta_1{}^2=\frac{p^2(1+m_1{}^2)}{(l+mm_1)^2} \qquad \text{............} \quad (18)$$

Similarly, if δ_2 be the length of the line intercepted between the new origin and the point of intersection of (11) with (14), we have

$$\delta_2{}^2=\frac{p^2(1+m_2{}^2)}{(l+mm_2)^2} \qquad \text{............} \quad (19)$$

Hence, from (18) and (19), we get

$$\delta_1{}^2\delta_2{}^2=\frac{p^4\left\{1+(m_1{}^2+m_2{}^2)+m_1{}^2m_2{}^2\right\}}{\left\{l^2+lm(m_1+m_2)+m^2.m_1m_2\right\}^2}.$$

Therefore, substituting for m_1, m_2 from the system of equations (15), (16), (17), we get

$$\delta_1\delta_2=\frac{p^2\sqrt{\left\{4h^2+(a-b)^2\right\}}}{am^2-2hml+bl^2}. \qquad \text{............} \quad (20)$$

But, if ϕ be the angle between the lines given by (10), we have

$$\tan\phi=\frac{2\sqrt{h^2-ab}}{a+b},$$

whence

$$\sin\phi=\frac{2\sqrt{h^2-ab}}{\sqrt{\left\{4h^2+(a-b)^2\right\}}},$$

so that the area of the triangle in question is

$$=\tfrac{1}{2}\,\delta_1\,\delta_2\sin\phi$$

$$=\frac{p^2\sqrt{h^2-ab}}{am^2-2hml+bl^2}$$

$$=\frac{\left\{l(hf-bg)+m(hg-af)+n(h^2-ab)\right\}^2}{(h^2-ab)^{\frac{3}{2}}\,(am^2-2hml+bl^2)},$$

by substituting for p from (12). Hence, finally, using the determinant notation, and altering the sign of n, we have the general

Theorem.—If the general equation of the second degree

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of right lines, the area of the triangle formed by this pair with the line

$$\lambda x+\mu y+\nu=0$$

is

$$\frac{\begin{vmatrix} a & h & \lambda \\ h & b & \mu \\ g & f & \nu \end{vmatrix}^2}{\begin{vmatrix} h & b \\ a & h \end{vmatrix}^{\frac{3}{2}} \begin{vmatrix} h & b & \mu \\ a & h & \lambda \\ \lambda & \mu & 0 \end{vmatrix}} \quad \ldots \text{(21)}$$

The length of the portion of $\lambda x+\mu y+\nu=0$ which is intercepted between the pair of lines is also easily found; for, from (12), the perpendicular from the point of intersection of the pair of lines on

$$\lambda x+\mu y+\nu=0$$

is at once seen to be

$$\frac{\begin{vmatrix} a & h & \lambda \\ h & b & \mu \\ g & f & \nu \end{vmatrix}}{\left\{(h^2-ab)(\lambda^2+\mu^2)^{\frac{1}{2}}\right\}} \quad \ldots\ldots\ldots\ldots \text{(22)}$$

Hence, the length of the intercepted portion is

$$2\,\frac{\begin{vmatrix} \lambda & -\mu \\ \mu & \lambda \end{vmatrix}^{\frac{1}{2}}}{\begin{vmatrix} h & b \\ a & h \end{vmatrix}^{\frac{1}{2}}} \cdot \frac{\begin{vmatrix} a & h & \lambda \\ h & b & \mu \\ g & f & \nu \end{vmatrix}}{\begin{vmatrix} h & b & \mu \\ a & h & \lambda \\ \lambda & \mu & 0 \end{vmatrix}} \quad \text{(23)}$$

The product of the two sides is, by a glance at (20), written down to be

$$\frac{\begin{vmatrix} 2h & a-b \\ b-a & 2h \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} a & h & \lambda \\ h & b & \mu \\ g & f & \nu \end{vmatrix}^2}{\begin{vmatrix} h & a \\ b & h \end{vmatrix}^2 \begin{vmatrix} h & b & \mu \\ a & h & \lambda \\ \lambda & \mu & 0 \end{vmatrix}} \quad \ldots \text{(24)}$$

As an application of the formula in (21), we can find the area of the parallelogram formed by the two lines

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

with

$$ax^2+2hxy+by^2=0$$

which are two lines through the origin parallel to the first pair. By subtracting the equations, we see that

$$2gx+2fy+c=0$$

represents that diagonal of the parallelogram which does not pass through the origin. The area of the triangle formed by this diagonal with the first pair is

$$\frac{\left\{2g\,(hf-bg)+2f\,(hg-af)-c\,(h^2-ab)\right\}^2}{4(h^2-ab)^{\frac{3}{2}}\,(af^2-2fgh+bg^2)},$$

and that formed with the second pair is

$$\frac{c^2\,(h^2-ab)^2}{4(h^2-ab)^{\frac{3}{2}}\,(af^2-2fgh+bg^2)}.$$

But, since the discriminant vanishes, it is clear that

$$2g(hf-bg)+2f\,(hg-af)-2c\,(h^2-ab)=0$$

$$af^2-2fgh+bg^2=c\,(ab-h^2).$$

Hence, adding the above expressions, the area of the quadrilateral in question is found to be

$$\frac{1}{2}\,\frac{c}{\sqrt{h^2-ab}}.$$

It may be noted that this expression is only apparently independent of f, g, for the vanishing of the discriminant shews that a, b, c, h are functions of f and g.

§§. 6—7. *The Circle.*

§. 6. **Meaning of the Constants in the Equation of a Circle.**— The equation of a circle

$$x^2+y^2+2gx+2fy+c=0$$

being thrown into the form

$$(x-g)^2+(y-f)^2=g^2+f^2-c,$$

the quantities $-g$, $-f$ are seen to be the coordinates of the centre, while, if r be the radius, we have

$$r^2=g^2+f^2-c.$$

To determine the geometric meaning of c, let δ be the distance of the centre from the origin, and t, either of the tangents drawn from the origin to the circle; then,

$$\delta^2=r^2+t^2$$

and, also,

$$\delta^2=f^2+g^2$$

$$r^2=f^2+g^2-c$$

which give

$$c=t^2. \qquad \ldots\ldots\ldots\ldots (25)$$

Hence, c denotes the square of the tangent drawn from the origin to the circle. We thus infer that, if the equations of a system of circles agree in either f or g, the locus of their centres is a right line parallel to a given line at a given distance from it, and their common chords are parallel, being all perpendicular to this given line; if both f and g are the same in all the equations, the system is concentric; if c alone is the same in all the equations, the circles are such as can be intersected orthogonally by a circle of radius $\sqrt{c}$, described round the origin as centre; and this shews at once that as a system of co-axal circles can be orthogonally intersected, their equations must necessarily be of the form

$$x^2+y^2-2kx = \pm\delta^2,$$

where δ is constant, but k variable.

The geometric meaning of c also furnishes the length of the tangent drawn from any point to a circle, for, the equation of the circle being

$$x^2+y^2+2gx+2fy+c=0,$$

and the point from which tangents are drawn being (x', y'), remove the origin to this point; then, the new absolute term is clearly the point-function of the circle with respect to the point (x', y'), and this, therefore, is the length of the tangent sought. It follows as a consequence of this, that the geometric meaning of the equation of the circle is that, if the length of the tangent drawn from any point to a circle vanishes, that point must be on the curve itself.

§. 7. **Chords and Tangents of Circles and Conics.**—The following equation of the chord joining the two points (x', y'), (x'', y''), on the circle

$$x^2+y^2=r^2 \qquad \text{............} \quad (26)$$

is due to Professor Burnside, (Salmon's *Conics*, §. 85, Ed. 1879, p. 80),

$$(x-x')(x-x'')+(y-y')(y-y'')=x^2+y^2-r^2. \text{............} \quad (27)$$

It is easily verified that this is actually the equation of the chord; the following geometrical interpretation, however, shews the genesis of the equation.

On the line joining the points (x', y'), (x'', y'') as diameter, describe a circle; any point (x, y) on this circumference is such that the lines joining (x, y), (x', y'), and (x, y), (x'', y''), include a right angle; this condition, expressed analytically, gives for the equation of the circle

$$(x-x')(x-x'')+(y-y')(y-y'')=0 \quad \text{............} \quad (28)$$

The chord in question may now be regarded as the common chord of the two circles represented by (26) and (28); and then, from the elementary principle that $S+kS'=0$ represents any locus through the common points of $S=0$, $S'=0$, we at once write down Burnside's equation (27), the proper value of k being easily seen to be given by

$$1+k=0.$$

The generalisation to the conic given by the general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{............} \quad (29)$$

is easy, *viz.*,

$$a(x-x')(x-x'')+2h(x-x')(y-y'')+b(y-y')(y-y'')=0 \quad (30)$$

represents any conic through (x', y'), (x'', y''), which may, it is useful to notice, satisfy three other conditions: and the chord in question, being the common chord of (29) and (30), must have for its equation

$$a(x-x')(x-x'')+2h(x-x')(y-y'')+b(y-y')(y-y'')$$
$$=ax^2+2hxy+by^2+2gx+2fy+c \quad \text{............} \quad (31)$$

I have not, however, been able to find if the conics (29) and (30) are connected by any very special or peculiar relation: their centres are not coincident; the centre of (30) is not on the chord whose equation is required; their asymptotes, however, include equal angles, and their axes are parallel; in fact, they are similar and similarly situated, and, therefore, necessarily equi-eccentric.

The equation of the tangent *at* any point may be deduced, as usual, from the equation of the chord; or we may first obtain by Joachimsthal's method the equation of the pair of tangents *from* an external point, and thence obtain the equation of the tangent *at* any point of the curve. The same equation, however, may be obtained by transformation, if we know the equation of the tangents from the origin; thus, the conic being

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and (x', y') the external point, remove the origin to this point, so that the conic becomes

$$ax^2+2hxy+by^2+2g'x+2f'y+c'=0,$$

where the values of f', g', c' are the same as in §. 4. If now $y=mx$ be any line through the new origin, it will touch the conic if the quadratic in x,

$$(a+2hm+bm^2)x^2+2(g'+f'm)x+c'=0,$$

has equal roots, which condition gives

$$c'(a+2hm+bm^2)=(g'+f'm)^2,$$

and by substituting

$$m=\frac{y}{x},$$

we have for the equation of the tangents, referred to the new origin,

$$c'(ax^2+2hxy+by^2)=(g'x+f'y)^2,$$

which may be written

$$c'(ax^2+2hxy+by^2+2g'x+2f'y+c')=(g'x+f'y+c')^2.$$

Reverting to our old axes, we have at once the equation in the form

$$(\text{Conic})\times(\text{Point-function})=(\text{Polar})^2,$$

which is, of course, the same equation as that obtained by Joachimsthal's method.

§§. 8—15. *The General Equation of the Second Degree.*

§. 8. **Preliminary.**—The discussion of the general equation of the second degree deservedly occupies an important position in the application of analytical geometry to the theory of lines of the second order; for, in analytical geometry properly so called, the question of degree or class is of fundamental importance, and the curves of the second degree should be called lines of the second order, and not conic sections, the proper point of view from which their properties ought to be studied being the fact that the equation representing them is of the second degree, and not the other fact that they are sections of a cone and have foci and directrices. The truly logical order of treating the subject is first to have a chapter on the equation of the first degree, containing the properties of right lines, then a chapter on the general equation of the second degree, and, as distinctly subsidiary to this, chapters on the circle, the ellipse, and the other conics. We proceed, then, to give the barest outline of such a systematic discussion as is indicated here. It may usefully be noted that the object of the discussion is twofold, *viz.*, in the first place, the problem is how to transform the equation to its simplest forms, and thus to classify the different kinds of conics; in the second place, we obtain some general formulæ for such properties as are common to all conics.

§. 9. **Transformation of the Equation.**—The general equation of the second degree being

$$S = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad \ldots\ldots\ldots\ldots \quad (32)$$

first change the origin to (x', y'), so that the equation becomes

$$ax^2 + 2hxy + by^2 + 2g'x + 2f'y + c' = 0$$

where

$$g' = \left(\frac{dS}{dx}\right)_{\substack{x=x'\\ y=y'}} = ax' + hy' + g$$

$$f' = \left(\frac{dS}{dy}\right)_{\substack{x=x'\\ y=y'}} = hx' + by' + f$$

$$c' = \text{Point-function.}$$

If, then, we make $g' = f' = 0$, that is, if we have for the coordinates of the new origin

$$x' = \frac{hf - bg}{ab - h^2}, \quad y' = \frac{hg - af}{ab - h^2}, \quad \ldots\ldots\ldots\ldots \quad (33), (34)$$

the transformed equation is

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0 \quad \ldots\ldots\ldots\ldots \quad (35)$$

where Δ is the discriminant (§. 4). In order that this transformation may be real and possible, we must have $(ab - h^2)$ different from zero.

The first point of departure, then, in the classification of conics, depends on the equation

$$ab - h^2 > \text{ or } < 0.$$

The case in which $h^2 = ab$ does not admit of the above transformation, and it must be treated separately (see Carr's Synopsis of Pure Mathematics, §§. 4430—4443). In the case where $(ab - h^2)$ does not vanish, we proceed further, as follows. Turn the axes about the new origin through an angle θ, where θ is given by

$$\tan 2\theta = \frac{2h}{a-b}, \quad \ldots\ldots\ldots\ldots (36)$$

and the new equation becomes

$$Ax^2 + By^2 + \frac{\Delta}{ab - h^2} = 0 \quad \ldots\ldots\ldots\ldots (37),$$

where A, B are certain constants to be determined hereafter. This equation may be put into the form

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \quad \ldots\ldots\ldots\ldots (38)$$

if

$$\frac{1}{\alpha^2} = -\frac{A}{Q}, \quad \frac{1}{\beta^2} = -\frac{B}{Q} \quad \ldots\ldots\ldots\ldots (39), (40)$$

and

$$Q = \frac{\Delta}{ab - h^2}. \quad \ldots\ldots\ldots\ldots (41)$$

Definition.—The quantity which we have denoted here by Q, we will call the **Asymptotic Constant,** the reason for which name will appear in §. 12. The quantities α, β are called the semi-axes of the conic.

§. 10. **Invariants.**—In the last section, we transformed the general equation of the second degree to its simplest form (38); but, we did not calculate the quantities α, β which depend on A, B. As a rule, the calculation of these quantities in every particular case is a laborious task; we, therefore, find out some functions of the coefficients which remain unaltered by transformation, and which are, accordingly, called Invariants of the conic. These invariants may be of different classes; thus, there are certain quantities which remain unaltered for a translation-transformation, and which may appropriately be called **Translation-invariants**; to this class belong a, h, b. Again, there are certain quantities which remain unaltered for a rotation-transformation, and which may, accordingly, be called **Rotation-invariants**; thus, the absolute term is a rotation-invariant; but the most important of these invariants are embodied in Dr. Boole's theorems that the quantities

$$\frac{a + b - 2h\cos\omega}{\sin^2\omega}, \quad \frac{ab - h^2}{\sin^2\omega} \quad \ldots\ldots\ldots\ldots (42), (43)$$

belong to this class (Salmon's *Conics*, §. 159, Ed. 1879, p. 159). Again, as we have seen that a, b, h are translation-invariants, it follows that

$$\frac{a+b-2h\cos\omega}{\sin^2\omega},\quad \frac{ab-h^2}{\sin^2\omega}$$

are invariants for the compound transformation as well, and may, accordingly, be called **General Invariants.** We shall now proceed to investigate, by a process analogous to that employed by Dr. Boole, certain invariants which include as particular cases those noticed above.

Suppose that by a rotation-transformation the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

assumes the form

$$AX^2+2HXY+BY^2+2GX+2FY+C=0.$$

Then, by the same transformation

$$x^2+y^2+2xy\cos\omega$$

is altered into

$$X^2+Y^2+2XY\cos\Omega,$$

because each of these expressions denotes the distance of the same point from the fixed origin. Hence, we have

$$(a+\lambda)x^2+2(h+\lambda\cos\omega)xy+(b+\lambda)y^2+2gx+2fy+c$$
$$=(A+\lambda)X^2+2(H+\lambda\cos\Omega)XY+(B+\lambda)Y^2+2GX+2FY+C.$$

Each side of this identity will resolve itself into linear factors for the same value of λ; hence, equating the discriminant of each side to zero, we have the two equations

$$c\sin^2\omega.\ \lambda^2+\left\{c(a+b-2h\cos\omega)-(f^2+g^2-2fg\cos\omega)\right\}\lambda$$
$$+abc+2fgh-af^2-bg^2-ch^2=0$$

$$C\sin^2\Omega.\ \lambda^2+\left\{C(A+B-2H\cos\Omega)-(F^2+G^2-2FG\cos\Omega)\right\}\lambda$$
$$+ABC+2FGH-AF^2-BG^2-CH^2=0.$$

As these quadratics in λ must be identical, we have, by equating the coefficients of corresponding terms, the two relations

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}-\frac{f^2+g^2-2fg\cos\omega}{c\sin^2\omega}$$
$$=\frac{A+B-2H\cos\Omega}{\sin^2\Omega}-\frac{F^2+G^2-2FG\cos\Omega}{C\sin^2\Omega},\ \ \ldots\ldots\ldots\ (44)$$

$$\frac{abc+2fgh-af^2-bg^2-ch^2}{c\sin^2\omega}=\frac{ABC+2FGH-AF^2-BG^2-CH^2}{C\sin^2\Omega}.\quad (45)$$

If $f=0$, $g=0$, these equations furnish Dr. Boole's invariants. As we have noticed that c is a rotation-invariant, these results shew that the functions

$$\left\{c(a+b-2h\cos\omega)-(f^2+g^2-2fg\cos\omega)\right\}\div\sin^2\omega\ \ldots\ldots\ldots\ (46)$$

$$\frac{\Delta}{\sin^2\omega} \qquad \ldots\ldots\ldots\ldots \text{(47)}$$

are rotation-invariants.

In order to see if any of these is a general invariant, we must examine whether they are translation-invariants. It will be found on examination that the first is not a translation-invariant, while for the second we know that, by a translation-transformation, the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is transformed into

$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0,$$

where

$$a'=a,\ h'=h,\ b'=b,$$

which, by the way, shews that the part of the second degree in the general equation is a covariant for translation-transformation, and

$$g'=ax'+hy'+g$$
$$f'=hx'+by'+f$$
$$c'=\text{Point-function},$$

from which, by actual calculation, we find that the coefficients of x^2, xy, y^2, x, y in

$$a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2$$

all vanish, and the absolute term is Δ. Hence, we infer that Δ is a translation-invariant, and so also is

$$\frac{\Delta}{\sin^2\omega},$$

since ω is unaltered by translation-transformation; thus, from what precedes, we have finally that

$$\frac{\Delta}{\sin^2\omega}$$

is a general invariant of the conic. To sum up, we enumerate below the principal invariants of the general conic.

I. Translation-invariants.

(i). a. (ii). h. (iii). b. (iv). Δ.

II. Rotation-invariants.

(i) Absolute term. (ii) $\dfrac{a+b-2h\cos\omega}{\sin^2\omega}$

(iii) $\dfrac{ab-h^2}{\sin^2\omega}$

(iv) $\dfrac{a+b-2h\cos\omega}{\sin^2\omega}-\dfrac{f^2+g^2-2fg\cos\omega}{c\sin^2\omega}$

(v) $\dfrac{\Delta}{\sin^2\omega}$. (vi) $\dfrac{f^2+g^2-2fg\cos\omega}{c\sin^2\omega}$.

III. General invariants.

(i) $\dfrac{a+b-2h\cos\omega}{\sin^2\omega}$ (ii) $\dfrac{ab-h^2}{\sin^2\omega}$

(iii) $\dfrac{\Delta}{\sin^2\omega}$.

It is clear that since any function of an invariant is an invariant, various invariants may be deduced from these by combining them in different ways or by imposing limiting conditions on them. Thus, for rectangular axes, Δ is a general invariant; and, if we examine the equation

$$ax^2+2hxy+by^2+2fy=0,$$

which denotes a conic referred to a tangent and normal as coordinate-axes, we see that it has the three general invariants, $(a+b)$, $(ab-h^2)$, af^2.

We have shewn above, *by actual calculation*, that the discriminant is a translation-invariant; it is interesting to note that the same result may be obtained as an illustration of Dr. Boole's method. Thus, if by translation-transformation the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is transformed into

$$a_1X^2+2h_1XY+b_1Y^2+2g_1X+2f_1Y+c_1=0,$$

the same transformation changes

$$x^2+y^2+2xy\cos\omega$$

into

$$(X-x_1)^2+(Y-y_1)^2+2(X-x_1)(Y-y_1)\cos\omega,$$

whence we have

$$\begin{aligned}&ax^2+2hxy+by^2+2gx+2fy+c+\lambda(x^2+y^2+2xy\cos\omega)\\&=a_1X^2+2h_1XY+b_1Y^2+2g_1X+2f_1Y+c_1\\&\quad+\lambda\left\{(X-x_1)^2+(Y-y_1)^2+2(X-x_1)(Y-y_1)\cos\omega\right\}.\end{aligned}$$

Equating the discriminant of the left hand side to zero, we have

$$c\sin^2\omega.\,\lambda^2+\left\{c(a+b-2h\cos\omega)-(af^2+bg^2-2fg\cos\omega)\right\}\lambda+\Delta=0 \quad\ldots\ldots\ldots\ldots (48)$$

If we equate to zero the discriminant of the right hand side, the equation in λ apparently comes out to be a cubic; but the coefficient of λ^3 is found on calculation to be zero, while, in the coefficient of λ^2, the terms involving x_1^2, x_1y_1, y_1^2, x_1, y_1 separately vanish, and the constant is $c\sin^2\omega$; hence the equation may be written

$$c\sin^2\omega.\,\lambda^2+R\lambda+\Delta_1=0. \quad\ldots\ldots\ldots\ldots (49).$$

Therefore, equating coefficients, we have

$$\Delta=\Delta_1,$$

which shews, as before, that Δ is a translation-invariant. It may be noted that, from a comparison of (48) and (49), it is clear that the value of R in (49) is

$$\left\{ c(a+b-2h\cos\omega)-(af^2+bg^2-2fg\cos\omega) \right\},$$

as, indeed, may be verified by direct calculation.

§. 11. **Lengths of axes and area of conic.**—We have shewn above that the semi-axes α, β of a conic are given by (39) and (40), *viz*,

$$\frac{1}{\alpha^2}=-\frac{A}{Q},\quad \frac{1}{\beta^2}=-\frac{B}{Q},$$

and, from the theory of invariants explained above, we have further

$$A+B=\frac{a+b-2h\cos\omega}{\sin^2\omega},\quad AB=\frac{ab-h^2}{\sin^2\omega}\quad \text{.........} \ (50),\ (51).$$

Hence, if ρ be a semi-axis, we have

$$\rho^4-(\alpha^2+\beta^2)\rho^2+\alpha^2\beta^2=0 \quad \text{...........} \ (52)$$

where

$$\alpha^2+\beta^2=-Q\left(\frac{1}{A}+\frac{1}{B}\right),\quad \alpha^2\beta^2=\frac{Q^2}{AB}.$$

Substituting in (52) from (50) and (51), and putting from (41)

$$Q=\frac{\Delta}{ab-h^2},$$

we get

$$\rho^4+\frac{\Delta(a+b-2h\cos\omega)}{(ab-h^2)^2}\rho^2+\frac{\Delta^2\sin^2\omega}{(ab-h^2)^3}=0,\quad \text{......} \ (53)$$

which is, accordingly, the equation furnishing the semi-axes of the given conic; and, as it is a quadratic in ρ^2, it shews that there are *four* semi-axes, which may be grouped into two pairs, the two axes in each pair being equal in magnitude but opposite in direction. It follows from (53) that, if $\rho_1{}^2$, $\rho_2{}^2$ be the roots of the quadratic in ρ^2, the area of the conic is

$$\pi\rho_1\rho_2=\frac{\pi\Delta\sin\omega}{(ab-h^2)^{\frac{3}{2}}} \quad \text{............} \ (54)$$

Again, it is clear that A and B will have the same sign or different signs, according as AB is positive or negative, that is, according as AB is greater or less than zero; hence, since A and B in the equation (37)

$$Ax^2+By^2+\frac{\Delta}{ab-h^2}=0$$

are connected by the relation (51)

$$AB=\frac{ab-h^2}{\sin^2\omega},$$

it follows that A and B, and thence necessarily α^2, β^2 in the equation (38)

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1,$$

will have the same sign or opposite signs, according as $(ab - h^2) \gt$ or $\lt 0$, or according as the curve is an ellipse or hyperbola. This completes the classification of conics. (§. 9).

§. 12. **Asymptotes.**—In the ordinary text-books (cf. Smith's *Conics*, §. 174, Ed. 1882, p. 187), the method of finding the equation of the asymptotes of the general conic is given as follows: it is first proved that the asymptotes of the conic in the particular case

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

and thence it is inferred that, in the general case, the equations of the conic and asymptotes must differ only by a constant; the logic of this reasoning is, to say the least, hardly satisfactory; the following method is both easy and rigorously logical.

The asymptotes being tangents to the conic at infinity, they may be regarded as a pair of lines passing through the points of intersection of the conic and the line at infinity. Now, the equation of the conic being

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

and that of the line at infinity having been shewn (§. 3) to be

$$\lambda = 0,$$

any conic through their common points is

$$S + \lambda = 0;$$

and, in order that this may be a pair of lines, its discriminant must vanish, whence, as usual,

$$\lambda = -Q = -\frac{\Delta}{ab - h^2},$$

and the asymptotes are given by

$$S = Q,$$

which shews that the **asymptotic constant** in (41) is a constant which must be equated to S, to furnish the equation of the asymptotes.

The above process may be represented in a modified form as follows; the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

being transformed to the centre, becomes

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0,$$

whence it at once follows that the quantity to be added to the right hand side of this equation to give the asymptotes is the asymptotic constant. Now, if we transform back to our old axes, the left hand side becomes

$$ax^2+2hxy+by^2+2gx+2fy+c,$$

while, Δ and $(ab-h^2)$ being translation-invariants, the right hand side remains unaltered, and the equation sought is accordingly

$$S=\frac{\Delta}{ab-h^2}. \qquad \text{............} \quad (55)$$

It follows from (6) that the point of intersection of the asymptotes in (55) coincides with the centre of the conic, and that, accordingly, the centre is the pole of the line at infinity. It is also clear that the asymptotes will be at right angles to each other and the conic will be a rectangular hyperbola, if $(a+b)=2h\cos\omega$, in oblique coordinates, and $(a+b)=0$ in rectangular coordinates.

§. 13. **Eccentricity.**—The eccentricity may be calculated in different ways according to the definition we employ.

First method.

$$e^2=\frac{\alpha^2-\beta^2}{\alpha^2},$$

where α, β are the semi-axes of the conic. We have

$$2-e^2=\frac{\alpha^2+\beta^2}{\alpha^2},$$

$$1-e^2=\frac{\beta^2}{\alpha^2},$$

which give

$$\frac{(2-e^2)^2}{1-e^2}=\frac{(\alpha^2+\beta^2)^2}{\alpha^2\beta^2},$$

and this, by substitution from (39) and (40), becomes

$$\frac{(2-e^2)^2}{1-e^2}=\frac{(A+B)^2}{AB}. \qquad \text{............} \quad (56)$$

But, from the invariants (42) and (43), we have

$$A+B=\frac{a+b-2h\cos\omega}{\sin^2\omega},$$

$$AB=\frac{ab-h^2}{\sin^2\omega},$$

so that equation (56) becomes

$$\frac{(2-e^2)^2}{1-e^2}=\frac{(a+b-2h\cos\omega)^2}{(ab-h^2)\sin^2\omega}, \qquad \text{............} \quad (57)$$

which is the familiar equation. It is clear from (57) that $(1-e^2)$ and $(ab-h^2)$ are simultaneously positive, zero, or negative; hence, we have

$$e^2 \angle = \urcorner\, 1$$

according as

$$h^2 \angle = \urcorner\, ab,$$

or according as the conic is an ellipse, a parabola, or an hyperbola. In the equilateral hyperbola, we have

$$a+b-2h\cos\omega=0,$$

whence $$e=\sqrt{2}.$$

Second method.

$$e=\sec\frac{\phi}{2},$$

where ϕ is the angle between the asymptotes. The equation of the asymptotes from (55) being

$$ax^2+2hxy+by^2+2gx+2fy+c=\frac{\Delta}{ab-h^2},$$

we have

$$\tan\phi=\frac{2\sin\omega.\sqrt{h^2-ab}}{a+b-2h\cos\omega}. \qquad \ldots\ldots\ldots\ldots (58)$$

But $$\sec^2\phi=\left(2\cos^2\frac{\phi}{2}-1\right)^{-2}$$

$$=\left[\frac{\sec^2\frac{\phi}{2}}{2-\sec^2\frac{\phi}{2}}\right]^2=\left(\frac{e^2}{2-e^2}\right)^2,$$

whence we have

$$\tan^2\phi=\sec^2\phi-1=\left(\frac{e^2}{2-e^2}\right)^2-1=\frac{4(e^2-1)}{(2-e^2)^2}.$$

Therefore, from equation (58),

$$\frac{e^2-1}{(2-e^2)^2}=\frac{(h^2-ab)\sin^2\omega}{(a+b-2h\cos\omega)^2},$$

which is the same equation as (57).

Third method.

The eccentricity may be defined to be the ratio of the distance of any point on the conic from a focus to its distance from the corresponding directrix; the calculation on the basis of this method will come in most appropriately when we presently deal with Laplace's Linear Equation of a Conic (§§. 16—20; see, in particular, §. 20).

§. 14. **Director-circle.**—The director-circle of

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

being the locus of intersection of orthogonal tangents, its equation in rectangular coordinates is known to be

$$(ab-h^2)(x^2+y^2)+2(gb-fh)x+2(fa-hg)y$$
$$+c(a+b)-f^2-g^2=0, \qquad \ldots\ldots\ldots\ldots (59)$$

which may also be written in the form

$$D\equiv(a+b)S-(ax+hy+g)^2-(hx+by+f)^2=0 \qquad \ldots\ldots\ldots\ldots (60)$$

The centre of the director-circle is seen from (59) to be the point

$$\left(\frac{fh-bg}{ab-h^2}, \frac{hg-af}{ab-h^2}\right),$$

which coincides with the centre of the conic; and, if R be the radius, we have

$$\begin{aligned} R^2 &= \frac{(fh-bg)^2}{(ab-h^2)^2}+\frac{(hg-af)^2}{(ab-h^2)^2}-\frac{c(a+b)-(f^2+g^2)}{ab-h^2} \\ &= \frac{-(a+b)\,\Delta}{(ab-h^2)^2}, \end{aligned}$$

which shews that in rectangular axes the square of the radius of the director-circle is equal to the sum of the squares of the semi-axes of the conic given in equation (53).

That the same propositions hold for oblique coordinates may easily be shewn, *viz.*, the equation of the tangents to the conic from (x', y') being

$$\begin{aligned} &(ax^2+2hxy+by^2+2gx+2fy+c)\times \\ &(ax'^2+2hx'y'+by'^2+2gx'+2fy'+c) \\ &= \left\{(ax'+hy'+g)x+(hx'+by'+f)y+gx'+fy'+c\right\}^2, \end{aligned}$$

the condition that these lines may include a right angle, gives for the locus of (x', y') the circle

$$\begin{aligned} &(ab-h^2)(x^2+y^2+2xy\cos\omega) \\ &+2\left\{(gb-fh)+(fa-gh)\cos\omega\right\}x \\ &+2\left\{(fa-gh)+(gb-fh)\cos\omega\right\}y \\ &+c(a+b)-(f^2+g^2)+2(fg-ch)\cos\omega=0 \end{aligned}$$

Comparing this with the standard form

$$(x-\alpha)^2+2(x-\alpha)(y-\beta)\cos\omega+(y-\beta)^2=r^2,$$

or

$$(x^2+y^2+2xy\cos\omega)-2(\alpha+\beta\cos\omega)x-2(\beta+\alpha\cos\omega)y \\ +\alpha^2+\beta^2+2\alpha\beta\cos\omega-r^2=0,$$

we have at once

$$\alpha=\frac{fh-bg}{ab-h^2},\quad \beta=\frac{hg-af}{ab-h^2},$$

which give the same coordinates of centre as before, while we have for the radius

$$\begin{aligned} r^2 &= \alpha^2+2\alpha\beta\cos\omega+\beta^2 \\ &\quad -\frac{c(a+b)-(f^2+g^2)+2(fg-ch)\cos\omega}{ab-h^2} \\ &= \Big[(fh-bg)^2+(hg-af)^2-(ab-h^2)\left\{c(a+b)-(f^2+g^2)\right\} \\ &\quad +2\left\{(fh-bg)(hg-af)-(fg-ch)(ab-h^2)\right\}\cos\omega\Big]\div(ab-h^2)^2 \end{aligned}$$

$$= \left[-(a+b)\,\Delta + 2h \cos \omega .\ \Delta \right] \div (ab-h^2)^2$$

$$= \frac{-(a+b-2h\cos\omega)\,\Delta}{(ab-h^2)^2},$$

which, by a glance at (53), is seen to represent, as before, the sum of the squares of the semi-axes. From the value of the radius given above, it is clear that, when the conic is an equilateral hyberbola, the radius vanishes, and the director-circle is a circle of infinitesimal radius, *viz.*, it is the centre of the conic itself, and the asymptotes, therefore, are the only tangents of the equilateral hyperbola which are at right angles to each other.

§. 15. **Hyperbola referred to the asymptotes.**—In this section, we purpose to investigate what form the general equation assumes when the axes of coordinates are transformed to the asymptotes; two methods will be given, the first very direct and elementary, the second partly geometrical and requiring a knowledge of the invariants given above.

First method.

Let the general equation of the second degree be

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Transfer the coordinate axes to the centre of the conic, which is also the point of intersection of the asymptotes; the conic then becomes

$$ax^2+2hxy+by^2+\frac{\Delta}{ab-h^2}=0 \quad \text{.........} \quad (61)$$

and the asymptotes are given by

$$ax^2+2hxy+by^2=0. \quad \text{............} \quad (62)$$

Now the equation of either asymptote may be taken to be $y=mx$, so that the two values of m are found, by substitution in (62), to be the roots of the quadratic

$$bm^2+2hm+a=0. \quad \text{............} \quad (63)$$

Hence, if α, β be the angles which the two asymptotes make with the axis of x, both $\tan\alpha$ and $\tan\beta$ must satisfy (63), so that we have

$$b\tan^2\alpha+2h\tan\alpha+a=0$$

or $$b\sin^2\alpha+2h\sin\alpha\cos\alpha+a\cos^2\alpha=0 \quad \text{......} \quad (64)$$

and similarly,

$$b\sin^2\beta+2h\sin\beta\cos\beta+a\cos^2\beta=0 \quad \text{............} \quad (65)$$

Now, the angle between the original axes being $\omega=\frac{\pi}{2}$, the ordinary formulæ for the transformation of coordinates (Salmon's *Conics*, §. 9, Ed. 1879, p. 7) become in this case

$$y\sin\omega = \mathrm{X}\sin\alpha+\mathrm{Y}\sin\beta.$$

$$x\sin\omega = \mathrm{X}\cos\alpha+\mathrm{Y}\cos\beta.$$

Substituting these in (61), and arranging, we have for the equation of the conic

$$(a\cos^2\alpha+2h\cos\alpha\sin\alpha+b\sin^2\alpha)X^2$$
$$+(a\cos^2\beta+2h\cos\beta\sin\beta+b\sin^2\beta)Y^2$$
$$+2\Big[a\cos\alpha\cos\beta+h.\sin(\alpha+\beta)+b\sin\alpha\sin\beta\Big]XY$$
$$+\frac{\Delta}{ab-h^2}=0.$$

But, by (64) and (65), the coefficients of X^2 and Y^2 vanish, and the equation becomes

$$2Hxy+\frac{\Delta}{ab-h^2}=0, \qquad \ldots\ldots\ldots\ldots (66)$$

where H is the quantity to be calculated. For this purpose, we note that, if m_1, m_2 be the two roots of the quadratic in m given by (63), we have

$$m_1+m_2=-\frac{2h}{b}, \quad m_1 m_2=\frac{a}{b}.$$

Now, we see that

$$H=\cos\alpha\cos\beta\left\{a+h(\tan\alpha+\tan\beta)+b\tan\alpha\tan\beta\right\}$$
$$=\frac{2(ab-h^2)}{b}\cos\alpha\cos\beta,$$

where

$$\cos^2\alpha\cos^2\beta=\left\{(1+m_1^2)(1+m_2^2)\right\}^{-1} \because m_1=\tan\alpha,\ m_2=\tan\beta.$$
$$=\Big[(m_1+m_2)^2+(1-m_1 m_2)^2\Big]^{-1}$$
$$=\frac{b^2}{(a-b)^2+4h^2}.$$

Therefore, $H=\pm\dfrac{2(ab-h^2)}{b}\cdot\dfrac{b}{\sqrt{\left[(a-b)^2+4h^2\right]}}$

and, finally, the equation (66) becomes

$$xy=\pm\frac{\Delta}{4}\cdot\frac{\sqrt{\left[(a-b)^2+4h^2\right]}}{(ab-h^2)^2}, \quad \ldots\ldots (67)$$

which is, accordingly, the equation of the hyperbola referred to its asymptotes, which was sought.

Second method.

The same result may also be obtained as follows. The equation of the conic, referred to its centre, being, as before,

$$ax^2+2hxy+by^2+\frac{\Delta}{ab-h^2}=0, \qquad \ldots\ldots\ldots\ldots (68)$$

and remembering that the absolute term is a rotation-invariant, we see that, when referred to the asymptotes, the equation must assume the form

$$Ax^2+2Hxy+By^2+\frac{\Delta}{ab-h^2}=0 \qquad \text{............ (69)}$$

Now, in this equation, the axis of x being an asymptote, one value of x must be infinite, and, therefore, in this equation, regarded as a quadratic in x, we must have $A=0$; similarly, the axis of y being the other asymptote, we must have $B=0$; so that (69) reduces to

$$2Hxy+\frac{\Delta}{ab-h^2}=0. \qquad \text{............ (70)}$$

To calculate H, we remark that, since the original axes are at right angles, we have $\omega=\frac{\pi}{2}$, and, as also $A=0$, $B=0$, the invariant relation

$$\frac{ab-h^2}{\sin^2\omega}=\frac{AB-H^2}{\sin^2\Omega}$$

reduces to

$$-H^2=(ab-h^2)\sin^2\Omega, \qquad \text{......... (71)}$$

where Ω is the angle between the asymptotes,

$$ax^2+2hxy+by^2=0. \qquad \text{............ (72)}$$

But, α, β being the angles which the asymptotes make with the axes, we have $\Omega=\alpha-\beta$, and, from equation (72),

$$\tan\Omega=\frac{2\sqrt{h^2-ab}}{a+b},$$

$$\sin\Omega=\frac{2\sqrt{h^2-ab}}{\sqrt{\left\{(a-b)^2+4h^2\right\}}},$$

so that (71) becomes

$$H^2=\frac{4(ab-h^2)^2}{(a-b)^2+4h^2},$$

and (70) gives for the required equation

$$xy=\pm\frac{\Delta}{4}\cdot\frac{\sqrt{\left\{(a-b)^2+4h^2\right\}}}{(ab-h^2)^2},$$

which is the same result as that obtained before. It may be noted that the value of H might have been obtained with equal ease by using the other invariant relation

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{A+B-2H\cos\Omega}{\sin^2\Omega}.$$

The geometrical meaning of this equation of the hyperbola is easily seen, *viz.*, taking ρ_1^2, ρ_2^2 for the squares of the semi-axes of the conic, and remembering that our original axes were rectangular, we have from (53),

$$\rho_1^2+\rho_2^2=\frac{-\Delta\,(a+b)}{(ab-h^2)^2}$$

$$\rho_1^2\,\rho_2^2=\frac{\Delta^2}{(ab-h^2)^3},$$

so that

$$(\rho_1^2-\rho_2^2)^2=(\rho_1^2+\rho_2^2)^2-4\,\rho_1^2\,\rho_2^2$$

$$=\frac{\Delta^2\left\{(a-b)^2+4h^2\right\}}{(ab-h^2)^4}.$$

The equation (67), therefore, may be written

$xy=\frac{1}{4}$ (Difference of squares of semi-axes),

which is a well-known result.

If the conic had been originally referred to axes inclined at an angle ω, the equation of the hyperbola referred to the asymptotes would have been

$$xy=\pm\frac{\Delta}{4\,(ab-h^2)^2}\left[(a-b)^2+4h^2-4\cos\omega\left\{h\,(a+b)-ab\cos\omega\right\}\right]^{\frac{1}{2}}$$

and the right hand side may be proved to be the difference of the squares of the semi-axes given by (53).

§§. 16—20. *Laplace's Linear Equation.*

§. 16. **Genesis of Laplace's Equation.**—The theorem that

$$\rho=Ax+By+C,$$

where ρ is the distance of any point on the curve from a fixed coplanar point, represents a conic is first due, substantially, to Laplace (*Mécanique Céleste*, Ed. 1878, t. I. p. 177). In integrating the equations for elliptic motion, he gets

$$dr=\lambda dx+\gamma dy,$$

which leads to

$$r=\frac{h^2}{\mu}+\lambda x+\gamma y\,;$$

Laplace then explicitly adds that "Cette équation, combinée avec celles-ci,

$$z=ax+by,\ r^2=x^2+y^2+z^2$$

donne une équation du second degré." It is proposed to examine here the geometrical meaning of the arbitrary constants in what I have called Laplace's Linear Equation to a conic.

§. 17. **Meaning of the Constants.**—That this equation represents a conic may be shewn in various ways, and some additional information regarding the constants may be gained from each standpoint of view. Thus, squaring the equation and putting

$$\rho^2=x^2+y^2,$$

we see that it is the equation to a conic which is an ellipse, a parabola, or an hyperbola according as

$$A^2+B^2 \angle = \rangle 1.$$

Now, knowing that the curve is a conic, we may next compare its equation with the focal polar equation

$$l=\rho\ (1+e \cos\theta).$$

Remembering that ρ is a function of x and y, we conclude that the absolute terms in the two equations must be identical, whence

$$C=l=\text{semi-latus-rectum.}$$

Again, as the equation may be written in the form

$$\rho \div \left\{\frac{Ax+By+C}{\sqrt{A^2+B^2}}\right\}=\sqrt{A^2+B^2},$$

where ρ is the distance of any point on the curve from a fixed point, and

$$\frac{Ax+By+C}{\sqrt{A^2+B^2}}$$

is the perpendicular on the line $Ax+By+C=0$, we see, by attending to the focus-directrix method of generating conics, that the curve is a conic of which the directrix is

$$Ax+By+C=0,$$

and the eccentricity is given by

$$e^2=A^2+B^2.$$

§. 18. **Elliptic Motion.**—In order to represent these properties geometrically, and to shew their relation to elliptic motion, it is convenient to begin with the following method of integrating the equations of motion. We have, as usual,

$$\frac{d^2x}{dt^2}=-\frac{\mu x}{r^3},$$

$$\frac{d^2y}{dt^2}=-\frac{\mu y}{r^3},$$

$$x\frac{dy}{dt}-y\frac{dx}{dt}=h,$$

$$r^2\frac{d\theta}{dt}=h.$$

Now $$\frac{x}{r}=\cos\theta,\ \frac{y}{r}=\sin\theta;$$

therefore $$\frac{d}{dt}\left(\frac{x}{r}\right)=-\sin\theta.\frac{d\theta}{dt}=-\frac{y}{r^3}.\ h,$$

whence $$\frac{y}{r^3}=-\frac{1}{h}\frac{d}{dt}\left(\frac{x}{r}\right),$$

and, similarly, $$\frac{x}{r^3}=\ \frac{1}{h}\frac{d}{dt}\left(\frac{y}{r}\right).$$

The equations of motion, therefore, become

$$\frac{d^2x}{dt^2} = -\frac{\mu}{h}\frac{d}{dt}\left(\frac{y}{r}\right),$$

$$\frac{d^2y}{dt^2} = \frac{\mu}{h}\frac{d}{dt}\left(\frac{x}{r}\right).$$

Integrating, we get

$$\frac{dx}{dt} = -\frac{\mu}{h}\left(\frac{y}{r} - \gamma\right),$$

$$\frac{dy}{dt} = \frac{\mu}{h}\left(\frac{x}{r} - \lambda\right),$$

and since

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h,$$

we have

$$\frac{\mu x}{h}\left(\frac{x}{r} - \lambda\right) + \frac{\mu y}{h}\left(\frac{y}{r} - \gamma\right) = h,$$

which leads to

$$r = \frac{h^2}{\mu} + \lambda x + \gamma y,$$

which is Laplace's equation. Comparing this with the form

$$\rho = Ax + By + C,$$

we find, as it ought to be,

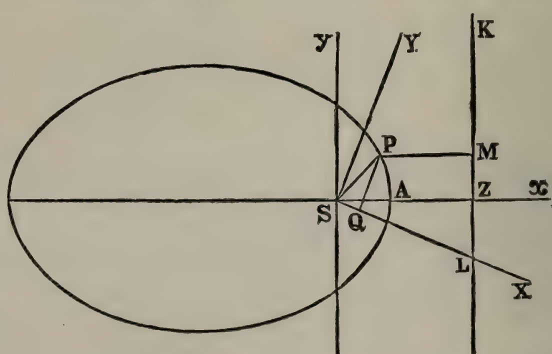
$$C = \frac{h^2}{\mu} = \text{semi-latus-rectum}.$$

This shews why, in integrating the equation

$$dr = \lambda dx + \gamma dy,$$

Laplace at once puts $\frac{h^2}{\mu}$ for the constant of integration.

§. 19. **Geometric interpretation.**—The subject may be made still clearer by the help of a diagram. The ellipse is originally referred to rectangular axes through the focus S; suppose that the coordinate axes revolve round the origin, making an angle XSx $(=\theta)$ with the former position. Then, we have



$$e.\ PM = PS,$$

whence

$$e^2.\ PM^2 = PS^2 = SQ^2 + QP^2$$
$$= x^2 + y^2.$$

But, as PM is parallel to SZ, we have

$$PM = p - x \cos\theta - y \sin\theta,$$

which gives

$$(ep - ex\cos\theta - ey\sin\theta)^2 = x^2 + y^2,$$

as might also have been obtained, but not so easily, by putting

$$x = X\cos\theta + Y\sin\theta$$
$$y = -X\sin\theta + Y\cos\theta$$

in the equation

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Comparing this with the equation

$$(C + Ax + By)^2 = \rho^2 = x^2 + y^2,$$

we get

$$C = ep,\ A = -e\cos\theta,\ B = -e\sin\theta,$$

whence, as before,

$$e^2 = A^2 + B^2.$$

Also $$\tan\theta = -\frac{B}{A},$$

and $$p = \frac{C}{e} = \frac{C}{\sqrt{A^2+B^2}}.$$

Now, when $\theta = 0$, the new axis of X coincides with the major axis of the ellipse; but, when $\theta = 0$, we have also $B = 0$, by virtue of the relation

$$\tan\theta = -\frac{B}{A};$$

therefore

$$(C + Ax)^2 = x^2 + y^2,$$

and, putting $x = 0$, this gives, as before,

$$y = C = \frac{h^2}{\mu}.$$

Again, the equation of the directrix is

$$x\cos\theta + y\sin\theta = p,$$

which, by substituting for θ and p, gives

$$Ax + By + C = 0,$$

and this agrees with our previous result.

It may be noticed that Gauss uses this form of the equation of a conic, and calls it the "*characteristic equation*" (*Theoria Motus*, §. 3). It is easy to see that when $B = 0$, we have $A = e$, and

$$\rho = C + ex,$$

which is the form finally adopted by Gauss. Since $x = \rho \cos \theta$, we have

$$\rho = \frac{C}{1 - e \cos \theta},$$

which is the ordinary polar equation. If $A = B = 0$, we have

$$\rho = \frac{h^2}{\mu},$$

which is the circle. The whole theory of lines of the second order may be based on the form

$$\rho = C + ex,$$

and, by means of this equation, Gauss has deduced the most complicated properties of elliptic motion with remarkable ease and elegance.

§. 20. **Eccentricity.**—If we square the equation

$$\rho = Ax + By + C,$$

and compare the result with the standard form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we have, by equating coefficients,

$$\frac{a}{c} = \frac{A^2 - 1}{C^2}, \quad \frac{h}{c} = \frac{AB}{C^2}, \quad \frac{b}{c} = \frac{B^2 - 1}{C^2}.$$

Therefore

$$\frac{(a-b)^2 + 4h^2}{c^2} = \frac{(A^2 - B^2)^2}{C^4} + \frac{4A^2B^2}{C^4} = \frac{(A^2 + B^2)^2}{C^4} = \frac{e^4}{C^4},$$

and

$$\frac{ab - h^2}{c^2} = \frac{(A^2 - 1)(B^2 - 1) - A^2B^2}{C^4} = \frac{1 - e^2}{C^4},$$

which lead to

$$e^4 + \frac{(a-b)^2 + 4h^2}{ab - h^2}(e^2 - 1) = 0,$$

and this is the well-known equation for the eccentricity (§. 13).

The value of the eccentricity in oblique axes may also be obtained from Laplace's equation; for, if p be the perpendicular on the directrix from any point on the curve

$$\rho = Ax + By + C,$$

we have

$$\rho = ep,$$

and

$$p = \frac{(Ax + By + C)\sin \omega}{\sqrt{A^2 + B^2 - 2AB\cos \omega}},$$

whence

$$e^2 = \frac{A^2 + B^2 - 2AB\cos \omega}{\sin^2 \omega}. \qquad \text{............ (73)}$$

Now, squaring Laplace's equation, and substituting for ρ^2, remembering that in oblique axes

$$\rho^2 = x^2 + y^2 + 2xy \cos \omega,$$

we get

$$(A^2-1)\,x^2+2\,(AB-\cos\omega)\,xy+(B^2-1)\,y^2+2ACx+2BCy+C^2=0,$$

a comparison of which with the standard equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

gives

$$\frac{a}{c}=\frac{A^2-1}{C^2},\ \frac{h}{c}=\frac{AB-\cos\omega}{C^2},\ \frac{b}{c}=\frac{B^2-1}{C^2},$$

whence

$$\frac{a+b-2h\cos\omega}{c}=\frac{A^2+B^2-2AB\cos\omega-2\sin^2\omega}{C^2}$$

$$=\frac{(e^2-2)\sin^2\omega}{C^2},$$

and

$$\frac{ab-h^2}{c^2}=\frac{\sin^2\omega-(A^2+B^2-2AB\cos\omega)}{C^4}$$

$$=\frac{(1-e^2)\sin^2\omega}{C^4},$$

by substitution from the value of e^2 in (73). These lead to the familiar result

$$\frac{(e^2-2)^2}{1-e^2}=\frac{(a+b-2h\cos\omega)^2}{(ab-h^2)\sin^2\omega}.$$

§. 21. *Area of a triangle.*

§. 21. **Triangle formed by two tangents.**—We now proceed to investigate the area of the triangle formed by two tangents drawn from any point to the general conic, and the chord of contact. For this purpose, we will first confine our attention to the simple case when the tangents are drawn from the origin, and then an easy application of invariants will smoothly lead to the solution of the general problem.

The tangents which can be drawn from the origin to the conic

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

are given by (Salmon's *Conics*, §. 147, Ed. 1879, p. 149)

$$(ac-g^2)\,x^2+2\,(ch-gf)\,xy+(bc-f^2)\,y^2=0,\quad\ldots\ (74)$$

and the chord of contact being the polar of the origin is

$$gx+fy+c=0.\quad\ldots\ldots\ldots\ldots\ (75)$$

The area of the triangle formed by the intersection of the lines in (74) and (75) is at once written down by substitution in (31), *viz.*,

$$(\text{Area})^2=\frac{c^3\,(af^2+bg^2+ch^2-2fgh-abc)}{af^2-2fgh+bg^2},$$

which may be written

$$\text{Area}=\frac{c\sqrt{-c\,\Delta}}{\left\{(ab-h^2)\,c-\Delta\right\}}\quad\ldots\ldots\ldots\ (76)$$

Now, if the tangents are drawn from any point (x', y') to the conic S, we may make that point our new origin, and by this transformation we know that c is changed into the point-function S′, while Δ and $(ab - h^2)$, being translation-invariants, remain unaltered by the transformation; hence, as a generalization of (76), we are able to enunciate the following general

Theorem.—If from any point (x', y'), tangents are drawn to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

the area of the triangle formed by the two tangents with their chord of contact is

$$\frac{S'\sqrt{-\Delta S'}}{(ab - h^2)S' - \Delta}, \quad \ldots\ldots\ldots\ldots (77)$$

where Δ is the discriminant and S′ the point-function of the conic.

A variety of particular theorems may be deduced from this general formula; thus, if the curve is a parabola, the area in question is

$$S'\sqrt{-\frac{S'}{\Delta}};$$

and, if, further, the point from which the tangents are drawn be the origin, we have the theorem that, if the general equation of the second degree represents a parabola, and two tangents be drawn from the origin to the curve, the area of the triangle formed by the two tangents and the chord of contact is

$$\frac{c\sqrt{c}}{\sqrt{a} - g\sqrt{b}}.$$

Again, the chord of contact being the polar of (x', y') with respect to the conic, has for its equation

$$(ax' + hy' + g)\,x + (hx' + by' + f)\,y + gx' + fy' + c = 0,$$

and, therefore, if p be the perpendicular let fall on this chord from (x', y'), we have easily

$$p^2 = \frac{S'^2}{(ax' + hy' + g)^2 + (hx' + by' + f)^2} \quad \ldots\ldots (78)$$

But, if $D \equiv 0$ be the equation of the director-circle of the conic, and, therefore, D′ its point-function, we have from (60)

$$(ax' + hy' + g)^2 + (hx' + by' + f)^2 = (a + b)\,S' - D'.$$

Hence (78) gives

$$p^2 = \frac{S'^2}{(a + b)\,S' - D'}. \quad \ldots\ldots\ldots\ldots (79).$$

It is now easy to find the length of the chord intercepted between the points of contact of the tangents, for if λ be the length sought, we have

$$\lambda = \frac{2(\text{Area of triangle})}{p},$$

which, by the help of equation (77), reduces to

$$\lambda = \frac{2\sqrt{\{\Delta\, S'\, D' - (a+b)\ \Delta\, S'^2\}}}{(ab-h^2)\ S' - \Delta}.$$

Hence, we have the

Theorem.—If from any point (x', y') two tangents be drawn to a conic given by the general equation, the length of the chord of contact is

$$\frac{2\sqrt{\{\Delta\, S'\, D' - (a+b)\ \Delta\, S'^2\}}}{(ab-h^2)\ S' - \Delta}, \qquad \ldots\ldots (80)$$

where S′, D′ are the point-functions of the conic and of its director-circle, respectively.

Various particular cases may be deduced from the general formula in (80). Thus, if the tangents be drawn from any point on the director-circle, that is, if the tangents be orthogonal, the length of the chord of contact is

$$\frac{2S'\sqrt{-(a+b)\ \Delta}}{(ab-h^2)\ S' - \Delta}.$$

Again, if two tangents be drawn from the directrix of a parabola to the curve, the length of the chord is

$$2S'\sqrt{-\frac{a+b}{\Delta}} = 2S'.\ \frac{\sqrt{a+b}}{f\sqrt{a} - g\sqrt{b}}.$$

If the curve is an equilateral hyperbola, the director-circle degenerates into the centre of the conic, and the chord in question, being the line at infinity, is of infinite length; this also follows from (80), for in this case

$$D' = 0,\ S' = \frac{\Delta}{ab-h^2},\ a+b = 0,$$

so that the numerator becomes the square root of a zero-quantity, while the denominator also vanishes, and, therefore, the limiting value of the apparently indeterminate expression is really infinite.

Again, we can easily find the area of the triangle formed by the chord of contact with the lines joining the centre to the points of contact. For the chord of contact, being the polar of (x', y'), is

$$(ax'+hy'+g)x + (hx'+by'+f)y + gx'+fy'+c = 0, \quad \ldots\ (81)$$

and the centre being

$$\left(\frac{hf-bg}{ab-h^2},\ \frac{hg-af}{ab-h^2}\right),$$

the perpendicular from the centre on the line in (81) is given by

$$\left\{ (ax'+hy'+g)(hf-bg)+(hx'+by'+f)(hg-af)+(gx'+fy'+c)(ab-h^2) \right\}$$
$$\div (ab-h^2)\left\{ (ax'+hy'+g)^2+(hx'+by'+f)^2 \right\}^{\frac{1}{2}}.$$

If, therefore, p_1 be the length of the perpendicular in question, this reduces to

$$p_1=\frac{\Delta}{(ab-h^2)\sqrt{\left\{(a+b)S'-D'\right\}}} \qquad \text{.........} (82)$$

Hence, as the length of the chord is given in (80), the area of the triangle is written down to be

$$\tfrac{1}{2}\,p_1\,\lambda=\frac{\Delta\sqrt{-\Delta S'}}{(ab-h^2)\left\{(ab-h^2)S'-\Delta\right\}} \qquad \text{.........} (83)$$

It must be carefully noticed that the two triangles whose areas are given in (77) and (83), being on opposite sides of the chord of contact, are affected with opposite signs; hence their algebraic sum establishes the truth of a property enunciated by Prof. Nash, *viz.*, we have the following

Theorem.—If two tangents are drawn from any point (x', y') to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

the area of the quadrilateral formed by the two tangents and the two lines joining the centre to the points of contact is

$$\frac{\sqrt{-\Delta S'}}{ab-h^2}, \qquad \text{............} (84)$$

where S' is the point-function of the conic.

It is easy to remark that the geometrical meaning of the equation of the conic is that, when the area of the quadrilateral vanishes, the locus of the point must be the curve itself. Again, since we know from geometry that the area of the quadrilateral is real or imaginary according as the point is outside or inside the curve, we infer from (84) that any given point is inside or outside the curve according as $\Delta S'$ is positive or negative, which is equivalent to the statement that the point is inside or outside according as the discriminant and the point-function have the same or different signs, and the same result, of course, also follows from the formula in (77). Here we may add that if from any point two tangents be drawn to a conic, the angle between the two tangents will be real, only if a certain relation holds amongst the coefficients in the equation of the conic; thus, first taking the simple case when the tangents are drawn from the origin, we have the tangents given by equation (74), *viz.*

$$(ac-g^2)x^2+2(ch-fg)xy+(bc-f^2)y^2=0,$$

and clearly the angle between these two lines will be real, if

$$(ch-fg)^2 > (ac-g^2)(bc-f^2)$$

or

$$\Delta < 0.$$

Hence, remembering that the discriminant is a translation-invariant, we can at once generalize the theorem to the case where the tangents are drawn from any point, *viz.*, the angle between the tangents is real, if the discriminant is negative; but we have shewn that, if the tangents are real and the point outside the curve, the discriminant and the point-function must have different signs, so that, as the discriminant is negative, the point-function must be positive; hence, finally, we have the very simple

Theorem.—Any point is outside a conic, on the curve, or inside it, according as the point-function is positive, zero, or negative.

§§. 22—23. *Inclinations of tangents to conics.*

§. 22. **Theorem.**—We shall now prove a theorem which shews how some well-known properties of the circle and the ellipse are correlated.

Consider the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots\ldots\ldots\ldots \ (85)$$

where b^2 is essentially indeterminate in sign and value. The tangents at any two points (x_1, y_1), (x_2, y_2) are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \qquad \ldots\ldots\ldots\ldots \ (86)$$

$$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1, \qquad \ldots\ldots\ldots\ldots \ (87)$$

and their chord of contact is

$$\frac{x\ (x_1+x_2)}{a^2}+\frac{y(y_1+y_2)}{b^2}=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+1 \qquad \ldots\ldots\ldots \ (88)$$

Hence, if θ, ϕ, ψ be the angles of inclination of the two tangents and of their chord of contact to a directrix, we have

$$\tan\theta=-\frac{a^2}{b^2}\cdot\frac{y_1}{x_1} \qquad \ldots\ldots\ldots\ldots \ (89)$$

$$\tan\phi=-\frac{a^2}{b^2}\cdot\frac{y_2}{x_2} \qquad \ldots\ldots\ldots\ldots \ (90)$$

$$\tan\psi=-\frac{a^2}{b^2}\cdot\frac{y_1+y_2}{x_1+x_2} \qquad \ldots\ldots\ldots\ldots \ (91)$$

Substituting for y_1, y_2 from (89) and (90) in

$$\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}=1,\ \ \frac{{x_2}^2}{a^2}+\frac{{y_2}^2}{b^2}=1,$$

we have

$$x_1=\frac{a^2}{\sqrt{a^2+b^2\tan^2\theta}},\quad x_2=\frac{a^2}{\sqrt{a^2+b^2\tan^2\phi}} \qquad \ldots\ldots \ (92),\ (93)$$

But, substituting for y_1, y_2 from (89) and (90) in (91), we have

$$\tan\psi = \frac{x_1 \tan\theta + x_2 \tan\phi}{x_1 + x_2} \quad \ldots\ldots\ldots\ldots \text{(94)}$$

Now, assume

$$a^2 + b^2 \tan^2\theta = \frac{a^2\lambda^2}{\cos^2\theta},$$

$$a^2 + b^2 \tan^2\phi = \frac{a^2\mu^2}{\cos^2\phi},$$

so that

$$\lambda^2 = 1 - e^2 \sin^2\theta,\ \mu^2 = 1 - e^2 \sin^2\phi,$$

and

$$x_1 = \frac{a\cos\theta}{\lambda},\ x_2 = \frac{a\cos\phi}{\mu}.$$

Substituting these values in (94), we arrive at the following symmetrical theorem, *viz.*, if θ, ϕ, ψ be the angles of inclination of any two tangents to a conic and of their chord of contact to a directrix, we have

$$\tan\psi = \frac{\lambda^{-1}\sin\theta + \mu^{-1}\sin\phi}{\lambda^{-1}\cos\theta + \mu^{-1}\cos\phi},$$

where the eccentricity of the conic is given by

$$e^2 = \frac{1-\lambda^2}{\sin^2\theta} = \frac{1-\mu^2}{\sin^2\phi}.$$

(See *Educational Times*, November 1885, my Ques. 8337).

§. 23. **Applications.**—To verify the truth of this theorem, we proceed to some applications. In the parabola, $e = 1$, so that

$$\lambda = \cos\theta,\ \mu = \cos\phi,$$

which give

$$2\tan\psi = \tan\theta + \tan\phi,$$

a result which can be proved independently, and is often useful in the elementary theory of projectiles. The particular case of the circle is specially interesting. Here $e = 0$, and $\lambda = \mu = 1$, whence

$$\tan\psi = \frac{\sin\theta + \sin\phi}{\cos\theta + \cos\phi} = \tan\frac{\theta+\phi}{2},$$

and

$$2\psi = \theta + \phi,$$

$$\text{or}\quad \psi - \theta = \phi - \psi.$$

To see the geometric meaning of this analytic condition, observe that, in the circle, the foci coincide with the centre, and the position of the axes becomes essentially indeterminate, while the directrix is situated at an infinite distance. Now draw any two tangents OA, OB to a circle, and let OA, OB, BA intersect the line at infinity in the points C, D, E; $\angle OCD = \theta$, $\angle ODC = -\phi$, $\angle BEC = \psi$, ϕ being taken negative as it is measured in a direction opposite to that in which θ, ψ are measured;

hence we have

$$\angle OAB = \angle CAE = \theta - \psi$$
$$\angle OBA = \psi - \phi.$$

Therefore $\angle OAB = \angle OBA$, and $OA = OB$, just as it should be, so that the geometric meaning is the equality of two tangents to a circle drawn from any external point. Lastly, if we draw any two tangents OA, OB to any conic, and, if OA, OB, BA intersect a directrix at C, D, E, we have as before

$$\angle OAB = \theta - \psi, \ \angle OBA = \psi - \phi.$$

Now draw through the centre two radii-vectores of the curve (ρ_1, ρ_2), making angles θ, ϕ with the conjugate axis; then, from the polar equation to the curve, we have

$$\rho_1^2 = \frac{b^2}{1 - e^2 \sin^2\theta}, \ \rho_2^2 = \frac{b^2}{1 - e^2 \sin^2\phi},$$

so that

$$\rho_1 = \frac{b}{\lambda}, \ \rho_2 = \frac{b}{\mu},$$

which furnish the geometrical meanings of the symbols λ, μ in the statement of the theorem. Substituting for λ, μ in our original equation, we have

$$\tan\psi = \frac{\rho_1 \sin\theta + \rho_2 \sin\phi}{\rho_1 \cos\theta + \rho_2 \cos\phi},$$

whence

$$\frac{\rho_1}{\rho_2} = \frac{\sin(\psi - \phi)}{\sin(\theta - \psi)} = \frac{OA}{OB},$$

and this asserts that the tangents OA, OB are proportional to the central radii-vectores which are obviously parallel to them. In the case of the circle, the indeterminateness in the position of the axes makes all the radii-vectores equal, so that, as shewn before,

$$OA = OB, \ \psi - \phi = \theta - \psi.$$

It may be remarked that we might have started from the polar instead of the Cartesian equations, as just shewn, and thus worked up to the value of $\tan\psi$ given above; it is also useful to notice that, though the theorem was obtained from a very particular form of the equation of a central conic, it is perfectly true for the general conic, inasmuch as the eccentricity only appears in the final result.

§. 24. *Similar Conics.*

§. 24. **Generation of Similar Conics.** Given any conic, any other conic which is concentric with it, and similar and similarly situated, may be generated as the locus of a point through which any two chords of the conic being drawn at right angles to each other, the sum of the reciprocals of the rectangles under the segments of each chord is constant, the variation of this constant furnishing the different members of the family of similar conics.

Let

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \ldots\ldots\ldots\ldots (95)$$

be the primitive conic, and (x', y') the point through which the chords are drawn at right angles to each other and whose locus we seek. Transferring the origin to this point, the conic becomes

$$ax^2+2hxy+by^2+2g'x+2f'y+c'=0 \qquad \ldots\ldots\ldots\ldots (96)$$

where c' is the point-function. The polar form of this equation is

$$(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)\,\rho^2+2(g'\cos\theta+f'\sin\theta)\,\rho+c'=0 \quad \ldots (97)$$

Hence, if ρ_1, ρ_2 be the segments of the chord drawn through the new origin, inclined at an angle θ to the axis of x, and ρ_3, ρ_4 the segments of the chord at right angles, we have, from (97),

$$\rho_1\,\rho_2=\frac{c'}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta},$$

$$\rho_3\,\rho_4=\frac{c'}{a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta}$$

so that

$$\frac{1}{\rho_1\,\rho_2}+\frac{1}{\rho_3\,\rho_4}=\frac{a+b}{c'},$$

which shews that the sum of the reciprocals of the rectangles is independent of the direction of the chord, and for any given value of this sum, say $\frac{1}{k^2}$, the locus of (x', y') is given by

$$\frac{a+b}{c'}=\frac{1}{k^2},$$

which may be written

$$ax^2+2hxy+by^2+2gx+2fy+c=k^2\,(a+b) \qquad \ldots\ldots\ldots\ldots (98)$$

and this, of course, represents a conic concentric with the primitive one given by (95), and similar and similarly situated; and we get a family of similar conics by assigning all possible values to k. It is interesting to remark that the property established here is general in a twofold sense, *viz.*, if the sum of the reciprocals of the rectangles under the segments is to be constant, the point may be any point on the conic given by (98), and the chords may be inclined at any angle to the axis of x, provided they include a right angle. The same results, of course, could have been obtained by applying the process to each of the conics separately, *viz.*, if we have the central conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the value of

$$\frac{1}{\rho_1 \rho_2}+\frac{1}{\rho_3 \rho_4}$$

is found to be

$$\frac{\dfrac{1}{a^2}+\dfrac{1}{b^2}}{\dfrac{x'^2}{a^2}+\dfrac{y'^2}{b^2}-1},$$

and the locus in question is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=k^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{k^2}\right).$$

Similarly, if we have the parabola

$$y^2=4ax,$$

the value of

$$\frac{1}{\rho_1 \rho_2}+\frac{1}{\rho_3 \rho_4}$$

is

$$\frac{1}{y'^2-4ax'},$$

and the locus sought is

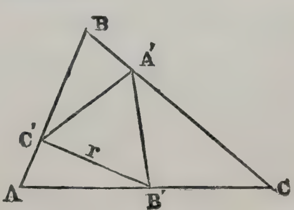
$$y^2-4ax=k^2.$$

Lastly, as in the equilateral hyperbola, we have $(a+b)=0$, the required conic-locus is the given conic itself, and we have the following

Theorem.—If through a given point P in the plane of any conic, any two chords be drawn mutually at right angles, the sum of the reciprocals of the rectangles under the segments is constant; and, for different values of this constant, the locus of P is a family of concentric, similar and similarly situated conics, which, however, all merge into the primitive conic when it is an equilateral hyperbola. (Cf. Salmon's *Conics*, §. 181, Ex. 2, Ed. 1879, p. 175).

§. 25. *Theory of Envelopes.*

§. 25. **On Three Parabolic Envelopes.**—As an illustration of the theory of envelopes, we proceed to discuss the envelopes of the sides of all equilateral triangles inscribed in a given triangle.



Let ABC be the given triangle, and A'B'C' an equilateral triangle inscribed in it; let r be the side of this equilateral triangle, and let $\angle AC'B'=\frac{\pi}{3}+\theta$, so that $\angle A'C'B=\frac{\pi}{3}-\theta$, $\angle BA'C'=\frac{2\pi}{3}+\theta-B$,

$\angle AB'C' = \frac{2\pi}{3} - \theta - A$. Then, in order to find the envelope of B′C′, take AC, AB as the axes of x and y respectively, so that the equation of B′C′ is

$$\frac{x}{AB'} + \frac{y}{AC'} = 1 \qquad \text{.............} (99)$$

Now, we have from the geometry of the figure

$$AC' = \frac{r}{\sin A} \sin\left(\frac{2\pi}{3} - \theta - A\right), \qquad \text{.............} (100)$$

$$AB' = \frac{r}{\sin A} \sin\left(\frac{\pi}{3} + \theta\right), \qquad \text{.............} (101)$$

while

$$c = AB = AC' + C'B$$

gives

$$\frac{c}{r} = \frac{\sin\left(\frac{2\pi}{3} - \theta - A\right)}{\sin A} + \frac{\sin\left(\frac{2\pi}{3} + \theta - B\right)}{\sin B}$$

$$= \left\{ \frac{\sin\left(\frac{2\pi}{3} - A\right)}{\sin A} + \frac{\sin\left(\frac{2\pi}{3} - B\right)}{\sin B} \right\} \cos\theta$$

$$+ \left\{ \frac{\cos\left(\frac{2\pi}{3} - B\right)}{\sin B} - \frac{\cos\left(\frac{2\pi}{3} - A\right)}{\sin A} \right\} \sin\theta,$$

which may be written in the form

$$\frac{c}{r} = P\cos\theta + Q\sin\theta, \qquad \text{.............} (102)$$

where

$$P = 1 + \frac{\sqrt{3}}{2}(\cot A + \cot B) \qquad \text{.............} (103)$$

$$Q = \tfrac{1}{2}(\cot A - \cot B) \qquad \text{.............} (104)$$

The equation of B′C′ in (99), therefore, reduces to

$$\frac{\sin A}{\sin\left(\frac{\pi}{3} + \theta\right)} x + \frac{\sin A}{\sin\left(\frac{2\pi}{3} - \theta - A\right)} y = r,$$

which may be written

$$\left\{ x\sin\left(\frac{2\pi}{3} - A\right) + y\sin\frac{\pi}{3} \right\} \cos\theta + \left\{ y\cos\frac{\pi}{3} - x\cos\left(\frac{2\pi}{3} - A\right) \right\} \sin\theta$$

$$= \frac{r}{\sin A} \cdot \sin\left(\frac{\pi}{3} + \theta\right) \sin\left(\frac{2\pi}{3} - \theta - A\right)$$

$$= \frac{r}{2\sin A} \left\{ \cos\left(\frac{\pi}{3} - 2\theta - A\right) + \cos A \right\},$$

and this may be written

$$r\left\{\cos A+\cos\left(\frac{\pi}{3}-A-2\theta\right)\right\}=E\cos\theta+F\sin\theta, \quad \ldots \quad (105)$$

where

$$E=2\sin A\left\{x\sin\left(\frac{2\pi}{3}-A\right)+y\sin\frac{\pi}{3}\right\} \quad \ldots\ldots\ldots\ldots \quad (106)$$

$$F=2\sin A\left\{y\cos\frac{\pi}{3}-x\cos\left(\frac{2\pi}{3}-A\right)\right\} \quad \ldots\ldots\ldots\ldots \quad (107)$$

Eliminating r between (102) and (105), we have

$$2c\left\{\cos A+\cos\left(\frac{\pi}{3}-A-2\theta\right)\right\}$$

$$=PE.\ 2\cos^2\theta+QF.\ 2\sin^2\theta+(QE+PF).\ 2\sin\theta\cos\theta.$$

Assuming, therefore, $2\theta=\phi$, this may be written

$$2c\cos A+2c\cos\left(\frac{\pi}{3}-A-\phi\right)$$

$$=PE+QF+(PE-QF)\cos\phi+(QE+PF)\sin\phi$$

Expanding $\cos\left(\frac{\pi}{3}-A-\phi\right)$, and arranging the coefficients of $\sin\phi$ and $\cos\phi$, this may be written

$$M\sin\phi+N\cos\phi=K \quad \ldots\ldots\ldots\ldots \quad (108)$$

where

$$M=QE+PF-2c\sin\left(\frac{\pi}{3}-A\right) \quad \ldots\ldots\ldots\ldots \quad (109)$$

$$N=PE-QF-2c\cos\left(\frac{\pi}{3}-A\right) \quad \ldots\ldots\ldots\ldots \quad (110)$$

$$K=2c\cos A-PE-QF \quad \ldots\ldots\ldots\ldots \quad (111)$$

The envelope of (108) is obviously

$$M^2+N^2=K^2,$$

and this, being written in the form

$$M^2=(K+N)(K-N),$$

leads, on substitution from (109), (110), and (111), to the equation

$$(QE+PF)^2-4c\sin\left(\frac{\pi}{3}-A\right).\ (QE+PF)+4c^2\sin^2\left(\frac{\pi}{3}-A\right)$$

$$=4c^2\left\{\cos^2 A-\cos^2\left(\frac{\pi}{3}-A\right)\right\}$$

$$-4c\left\{\left[\cos A-\cos\left(\frac{\pi}{3}-A\right)\right]PE+\left[\cos A+\cos\left(\frac{\pi}{3}-A\right)\right]QF\right\}$$

$$+4\ PQEF,$$

which may be written

$$(\mathrm{QE}-\mathrm{PF})^2-4c\left[\mathrm{Q}.\sin\left(\frac{\pi}{3}-\mathrm{A}\right)+\mathrm{P}\left\{\cos\left(\frac{\pi}{3}-\mathrm{A}\right)-\cos\mathrm{A}\right\}\right]\mathrm{E}$$

$$-4c\left[\mathrm{P}\sin\left(\frac{\pi}{3}-\mathrm{A}\right)-\mathrm{Q}\left\{\cos\left(\frac{\pi}{3}-\mathrm{A}\right)+\cos\mathrm{A}\right\}\right]\mathrm{F}+4c^2\sin^2\mathrm{A}=0.$$

As E and F are linear functions of x and y, while P and Q are constant quantities, it is clear that this equation of the required envelope represents a parabola, and a diameter of this parabolic envelope is given by

$$\mathrm{QE}=\mathrm{PF},$$

which is equivalent to

$$\left\{\mathrm{P}.\cos\left(\frac{2\pi}{3}-\mathrm{A}\right)+\mathrm{Q}\sin\left(\frac{2\pi}{3}-\mathrm{A}\right)\right\}x-\left\{\mathrm{P}\cos\frac{\pi}{3}-\mathrm{Q}\sin\frac{\pi}{3}\right\}y=0,$$

or, since

$$\mathrm{P}\cos\left(\frac{2\pi}{3}-\mathrm{A}\right)+\mathrm{Q}\sin\left(\frac{2\pi}{3}-\mathrm{A}\right)=\frac{\sin\left(\frac{\pi}{3}+\mathrm{C}\right)}{\sin\mathrm{B}},$$

and

$$\mathrm{P}\cos\frac{\pi}{3}-\mathrm{Q}\sin\frac{\pi}{3}=\frac{\sin\left(\frac{\pi}{3}+\mathrm{B}\right)}{\sin\mathrm{B}},$$

the equation of the diameter may be written

$$x\sin\left(\frac{\pi}{3}+\mathrm{C}\right)-y\sin\left(\frac{\pi}{3}+\mathrm{B}\right)=0.$$

The diameter can be geometrically constructed as follows, *viz.*, on BC describe externally an equilateral triangle BDC, and join AD; then AD is the diameter; for, if the point D be (x, y), we have

$$\frac{\mathrm{DC}}{y}=\frac{\sin\mathrm{A}}{\sin\left(\frac{\pi}{3}+\mathrm{C}\right)},\quad\frac{\mathrm{DB}}{x}=\frac{\sin\mathrm{A}}{\sin\left(\frac{\pi}{3}+\mathrm{B}\right)},$$

so that the equation of AD is

$$x\sin\left(\frac{\pi}{3}+\mathrm{C}\right)=y\sin\left(\frac{\pi}{3}+\mathrm{B}\right),$$

which is also the equation of the diameter.

Again, if we consider the envelopes of the other two sides, they also will be parabolas, and their diameters will be obtained by joining B and C to the remote vertices E and F of the equilateral triangles described externally on the opposite sides; and, since, from elementary geometry, AD, BE, CF intersect in a point, it can easily be shewn, from Euc. III. 22, that the acute angle between any two of them is $\frac{\pi}{3}$. Thus, finally, we have the

Theorems.—The envelopes of the sides of the equilateral triangles which can be inscribed in any given triangle ABC, are three parabolas;

the acute angle between every pair of the three axes is $\frac{\pi}{3}$; if, through the vertices of the given triangle, diameters of the parabolas be drawn, they intersect in a fixed point which may be determined geometrically, *viz.*, if equilateral triangles BDC, CEA, AFB be described externally on the sides, the lines AD, BE, CF are diameters of the enveloping parabolas and meet in a point, the acute angle between each pair being $\frac{\pi}{3}$.

§§. 26—27. *Reciprocal Polars.*

§. 26. **Reciprocal of Central Conic.**—It is well-known that the first focal pedal of a conic, being the locus of the foot of the perpendicular dropped from a focus on any tangent, is, in the case of central conics, the circle described on the axis-major as diameter ; hence, as the reciprocal of any curve is the inverse of its pedal, it is clear that the inverse of pedal of the first focal pedal of any central conic is the reciprocal polar of a circle, which reciprocal is known to be a conic ; hence it follows that the second pedal of a conic with respect to a focus is the inverse of a conic whose position and magnitude may be determined geometrically. For we know that the reciprocal of a circle of radius a, with respect to a circle of radius k, is a conic which is an ellipse if the origin of reciprocation lies within the given circle, the focus of the conic is at the origin of reciprocation, the semi-latus-rectum is $\frac{k^2}{a}$, the eccentricity is $\frac{c}{a}$, where c is the distance between the centres of the given circle and the circle of reciprocation, and the directrix is a line at right angles to the central line drawn at a distance $\frac{k^2}{c}$ from the origin of reciprocation. Now, in the question under consideration, we have to find the reciprocal of the circle described on the major axis as diameter, with a focus as origin of reciprocation; hence the conic is an ellipse, a focus of which is the focus of the given conic, the semi-latus-rectum is $\frac{k^2}{a}$, the eccentricity is equal to the eccentricity of the given conic, and the directrix is a line at right angles to the axis-major of the given conic, at a distance $\frac{k^2}{ae}$ from the given focus.

These results are easily verified analytically, for the given conic being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

remove the origin to the focus, say the negative one ; then the conic is

$$\frac{(x-ae)^2}{a^2}+\frac{y^2}{b^2}=1,$$

and the first pedal, being the circle on the major axis as diameter, is

$$(x-ae)^2+y^2=a^2,$$

the coordinates of any point on which may be expressed by means of a single parameter, *viz.*,

$$x=a\ (e+\cos\phi),$$
$$y=a\sin\phi,$$

and hence the equation of any tangent may be thrown into the form

$$(x-ae)\cos\phi+y\sin\phi=a.$$

A line at right-angles to this through the origin (which is now the focus) is

$$x\sin\phi-y\cos\phi=0,$$

and, as the second pedal of the conic, or the first pedal of the circle, is the locus of the intersection of the two lines, we have, by solving for $\sin\phi$ and $\cos\phi$,

$$\sin\phi=\frac{ay}{x^2+y^2-aex},\ \cos\phi=\frac{ax}{x^2+y^2-aex},$$

where (x, y) is, of course, a point on the pedal, *viz.*, the actual equation is

$$a^2\ (x^2+y^2)=(x^2+y^2-aex)^2,$$

which quartic, therefore, is the second pedal of the given conic with respect to a focus. To see that this is the inverse of a conic, we have only to take its inverse, *viz.*, substituting for x and y

$$\frac{k^2x}{x^2+y^2},\ \frac{k^2y}{x^2+y^2}$$

respectively, the second-pedal-quartic is seen to be the inverse of

$$a^2\ (x^2+y^2)=(k^2-aex)^2,$$

which is, of course, a conic, *viz.*, this may be written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^4}{a^2b^2}-\frac{2k^2ex}{ab^2},$$

which is equivalent to

$$\left(\frac{x}{a}-\frac{k^2e}{b^2}\right)^2+\frac{y^2}{b^2}=\frac{k^4}{b^4}.$$

It may be noted that any two conics having a common focus have two of their common chords passing through the intersection of their directrices; in the present case, therefore, two of the chords of intersection of this conic and the given conic are parallel to the directrices; one of these chords is found, by subtracting the equations of the conics, to be the line

$$x=\frac{k^2-b^2}{2ae}.$$

§. 27. **Reciprocal of Evolute of Conic.***—We now purpose to investigate the reciprocal polars of evolutes of conics; but as all central conics are included in the equation

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1, \qquad \text{............ (112)}$$

we will discuss the problem with regard to this general case. Since the reciprocal is the inverse of the pedal, and as the pedal of the evolute is the locus of the intersection of the normal and the line drawn at right angles to it through the origin, it is clear that the reciprocal polar of the evolute is the inverse of the locus of the point of intersection of the normal at any point of the curve, and the right line dropped perpendicular to it from the origin. Now, the normal at any point (x, y) of the curve in (112) is

$$\frac{m}{a}\left(\frac{x}{a}\right)^{m-1}(Y-y) = \frac{m}{b}\left(\frac{y}{b}\right)^{m-1}(X-x),$$

where X, Y being the current coordinates, the equation may be written

$$\frac{x^{m-1}}{a^m}Y - \frac{y^{m-1}}{b^m}X = xy\left(\frac{x^{m-2}}{a^m} - \frac{y^{m-2}}{b^m}\right). \qquad \text{......... (113)}$$

The straight line through the origin at right angles to this, is

$$\frac{y^{m-1}}{b^m}Y + \frac{x^{m-1}}{a^m}X = 0 \qquad \text{............ (114)}$$

At the common point of intersection of the two lines given by (113) and (114), we have

$$\left\{\frac{x^{2(m-1)}}{a^{2m}} + \frac{y^{2(m-1)}}{b^{2m}}\right\}X = -xy\left(\frac{y^{m-1}}{b^m}\right)\left(\frac{x^{m-2}}{a^m} - \frac{y^{m-2}}{b^m}\right) \quad (115)$$

$$\left\{\frac{x^{2(m-1)}}{a^{2m}} + \frac{y^{2(m-1)}}{b^{2m}}\right\}Y = xy\left(\frac{x^{m-1}}{a^m}\right)\left(\frac{x^{m-2}}{a^m} - \frac{y^{m-2}}{b^m}\right). \quad (116)$$

If (ξ, η) be the inverse of the point whose coordinates are given by (115) and (116), and k^2 the constant of inversion, we have

$$\xi = \frac{k^2X}{X^2+Y^2} = -\frac{k^2y^{m-1}}{xy.\, b^m\left(\dfrac{x^{m-2}}{a^m} - \dfrac{y^{m-2}}{b^m}\right)}. \qquad \text{......... (117)}$$

* The theorems established in this section were discovered by me about three years ago, and were, on the 29th August, 1885, communicated to Mr. W. J. C. Miller, Mathematical Editor of the *Educational Times*, with a view to their publication in that journal. They have since been published as questions 8571, 8707, 8773, 8993, 9049, 9074, 9148, 9162, 9163, 9204; but, while some of these questions have appeared under my name, the others have been, for reasons best known to Mr. Miller himself, ascribed to different gentlemen who had, perhaps, just as much to do with the theorems with which they have been credited, as the proverbial man in the moon.

$$\eta = \frac{k^2 Y}{X^2+Y^2} = \frac{k^2 . x^{m-1}}{xy . a^m \left(\frac{x^{m-2}}{a^m} - \frac{y^{m-2}}{b^m}\right)} \qquad \ldots\ldots\ldots\ldots (118)$$

If, now, we eliminate x and y between the equations (117) and (118) by virtue of the relation

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1,$$

we shall obtain the equation of the locus sought. For this purpose, we find that

$$\left(\frac{\xi}{+a}\right)^{\frac{m}{m-1}} + \left(\frac{\eta}{-b}\right)^{\frac{m}{m-1}}$$

$$= \left\{\frac{-k^2}{abxy\left[\frac{x^{m-2}}{a^m} - \frac{y^{m-2}}{b^m}\right]}\right\}^{\frac{m}{m-1}} \qquad \ldots\ldots\ldots\ldots (119)$$

and

$$(b\eta)\left(\frac{\xi}{+a}\right)^{\frac{1}{m-1}} + (a\xi)\left(\frac{\eta}{-b}\right)^{\frac{1}{m-1}}$$

$$= k^2 \left\{\frac{-k^2}{abxy\left[\frac{x^{m-2}}{a^m} - \frac{y^{m-2}}{b^m}\right]}\right\}^{\frac{1}{m-1}} \qquad \ldots\ldots\ldots\ldots (120)$$

Therefore, finally, replacing (ξ, η) by (x, y), we find from (119) and (120) the

Theorem.—The reciprocal polar of the evolute of

$$\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1$$

is the curve

$$k^2 \left\{\left(\frac{x}{+a}\right)^{\frac{m}{m-1}} + \left(\frac{y}{-b}\right)^{\frac{m}{m-1}}\right\}^{\frac{1}{m}}$$

$$= (by)\left(\frac{x}{+a}\right)^{\frac{1}{m-1}} + (ax)\left(\frac{y}{-b}\right)^{\frac{1}{m-1}}, \qquad \ldots\ldots\ldots\ldots (121)$$

where k is the radius of the circle of inversion.

A host of interesting results may be obtained by assigning particular values to m and k in (121); a few are noted below.

If $m=2$, $k^2=a^2\mp b^2$, we see that the reciprocal polar of the evolute of the conic

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1,$$

with regard to the circle described on the line joining the foci as diameter, is the curve

$$\frac{a^2}{x^2}\pm\frac{b^2}{y^2}=1 \qquad \text{............ (122)}$$

which, when the hyperbola is equilateral, becomes

$$\frac{1}{x^2}-\frac{1}{y^2}=\frac{1}{a^2}. \qquad \text{............ (123)}$$

Again, if $m=\frac{2}{3}$, $k=1$, we see that the reciprocal polar of the evolute of the hypocycloid

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}}+\left(\frac{y}{\beta}\right)^{\frac{2}{3}}=1$$

is the curve

$$\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}\right)^3=\left(\frac{x^4}{\alpha^2}-\frac{y^4}{\beta^2}\right)^2, \qquad \text{............ (124)}$$

the radius of the circle of inversion being unity; if $\alpha=\beta$, the polar equation of the reciprocal polar becomes

$$r=\alpha\sec 2\theta. \qquad \text{............ (125)}$$

Again, since the evolute of the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$\left(\frac{x}{\alpha}\right)^{\frac{2}{3}}+\left(\frac{y}{\beta}\right)^{\frac{2}{3}}=1,$$

where

$$\alpha=\frac{a^2-b^2}{a},\ \beta=\frac{a^2-b^2}{b},$$

we see, by putting $m=\frac{2}{3}$, $k^2=a^2-b^2$, that the reciprocal polar of the evolute of the evolute of the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with respect to the circle described on the line joining the foci as diameter, is the curve

$$\left(\frac{a^2}{x^2}+\frac{b^2}{y^2}\right)^3=(a^2x-b^2y)^2 \qquad \text{............ (126)}$$

Again, by putting $m=-2$, and attending to equation (122), it is clear that the reciprocal polar of the evolute of the reciprocal polar of the evolute of the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

with regard to the circle described on the line joining the foci as diameter, is the curve

$$\left\{\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}\right\}\left\{\left(\frac{x}{y}\right)^{\frac{1}{3}}\left(\frac{x}{b}\right)^{\frac{2}{3}}-\left(\frac{y}{x}\right)^{\frac{1}{3}}\left(\frac{y}{a}\right)^{\frac{2}{3}}\right\}^2=\left(\frac{a}{b}-\frac{b}{a}\right)^2 \quad (127)$$

Here we may remark in passing that since the reciprocal polar of the evolute of the reciprocal polar of any curve can be geometrically proved to be the locus of the extremity of the polar subtangent, it is clear that the curve in (127) is the locus of the extremity of the polar subtangent of the evolute of the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Hence, transforming to polar coordinates, we have the

Theorem.—The locus of the extremity of the polar subtangent of the curve

$$\left(\frac{1}{r}\right)^{\frac{2}{3}}=\left(\frac{a\cos\theta}{a^2\cdot\ b^2}\right)^{\frac{2}{3}}+\left(\frac{b\sin\theta}{a^2-b^2}\right)^{\frac{2}{3}},$$

which is, of course, the evolute of the conic, is the curve

$$\left(\frac{a}{b}-\frac{b}{a}\right)^2\frac{1}{r^2}=\left\{\left(\frac{\cos\theta}{a}\right)^{\frac{2}{3}}+\left(\frac{\sin\theta}{b}\right)^{\frac{2}{3}}\right\}\times$$

$$\left\{\cot\theta.\left(\frac{\cos\theta}{b}\right)^{\frac{2}{3}}-\tan\theta\left(\frac{\sin\theta}{a}\right)^{\frac{2}{3}}\right\}^2 \quad \ldots\ldots\ldots\ldots (128)$$

which is, of course, the polar form of the equation (127).

Again, by putting $m=\frac{1}{2}$, $k^2=ab$, we find that the reciprocal polar of the evolute of the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}=1,$$

with respect to a circle of radius $\sqrt{ab}$, is the cubic curve

$$\frac{y}{x}.\frac{a-y}{b}+\frac{x}{y}.\frac{b-x}{a}=2. \quad \ldots\ldots\ldots\ldots (129)$$

By the application of the same process to the parabola, a variety of new theorems may be obtained, *viz.*, taking the parabola of the n^{th} degree,

$$y=\lambda x^n, \quad \ldots\ldots\ldots\ldots (132)$$

the normal at any point (x, y) is

$$\lambda nx^{n-1}.\,Y+X=x\left(1+\lambda nyx^{n-2}\right), \quad \ldots\ldots\ldots\ldots (131)$$

while the line at right angles to this through the origin is

$$Y - \lambda n x^{n-1} X = 0, \qquad \text{............ (132)}$$

so that, at the point of intersection of the lines given by (131) and (132), we have

$$\left\{1 + \lambda^2 n^2 x^{2(n-1)}\right\} Y = \lambda n x^{n-1}.\, x.\, (1 + \lambda n y x^{n-2}), \qquad \text{...... (133)}$$

$$\left\{1 + \lambda^2 n^2 x^{2(n-1)}\right\} X = x\, (1 + \lambda n y x^{n-2}), \qquad \text{...... (134)}$$

and the inverse of the X, Y is given by

$$\xi = \frac{k^2 X}{X^2 + Y^2} = \frac{k^2}{x\,(1 + \lambda n y x^{n-2})} \qquad \text{............ (135)}$$

$$\eta = \frac{k^2 Y}{X^2 + Y^2} = \frac{k^2.\, \lambda n x^{n-1}}{x\,(1 + \lambda n y x^{n-2})} \qquad \text{............ (136)}$$

where ξ, η are the coordinates of a point on the locus sought; hence, eliminating x, y between the equations (135) and (136), by virtue of the relation in (130), we have, after replacing ξ, η by x, y respectively, the

Theorem.—The reciprocal polar of the evolute of the parabola of the n^{th} degree

$$y = \lambda x^n$$

is the curve

$$y x^{n-2} \left(1 + \frac{1}{n}.\frac{y^2}{x^2}\right)^{n-1} = \lambda n k^{2(n-1)} \qquad \text{............ (137)}$$

where k is the constant of inversion.

As before, by assigning particular values to λ and n in this equation, we may deduce various theorems.

Thus, the reciprocal polar of the evolute of the parabola

$$y^2 = 4ax,$$

with regard to a circle whose diameter is equal to the latus-rectum, is the cubic curve

$$r\,(\cos^2\theta + \cot^2\theta) = 4a \cos\theta, \qquad \text{............ (138)}$$

of which $x = 2a$ is an asymptote.

Again, the reciprocal polar of the evolute of the parabola

$$y^2 = 4ax,$$

with respect to a circle of radius a, is the cubic

$$x^3 = y^2\,(a - 2x), \qquad \text{............ (139)}$$

of which $x = \frac{a}{2}$ is an asymptote.

Again, the reciprocal polar of the evolute of the parabola

$$y^2 = 4a\,(x + a),$$

the focus being now the origin, with regard to a circle whose diameter is equal to the semi-latus-rectum, is the curve

$$r \cot\theta = a \sin\theta, \qquad \text{........... (140)}$$

which represents a circular cubic, of which $x=a$ is an asymptote, and the point at infinity a point of inflexion.

Again, the reciprocal polar of the evolute of the evolute of the parabola

$$y^2=4a\ (x+2a),$$

the origin now being the centre of curvature at the vertex, with respect to a circle of radius a, is the quartic

$$y^2\ (3x^2+2y^2)=a^3\,x^3. \qquad \ldots\ldots\ldots\ldots (141)$$

Similarly, the reciprocal polar of the evolute of the parabola

$$y^2=4a\ (x+2a),$$

with respect to a circle of radius k, is the cubic

$$ax^3=k^2y^2.$$

It is useful to notice that if we are given any curve

$$u=f\ (x,\ y)=0, \qquad \ldots\ldots\ldots\ldots (142)$$

the normal at any point $(x,\ y)$ is

$$(\mathrm{Y}-y)\ \frac{du}{dx}=(\mathrm{X}-x)\ \frac{du}{dy}, \qquad \ldots\ldots\ldots\ldots (143)$$

while the line at right angles to this through the origin is

$$\mathrm{X}\ \frac{du}{dx}+\mathrm{Y}\ \frac{du}{dy}=0. \qquad \ldots\ldots\ldots\ldots (144)$$

At the common point of intersection of these two lines, we have

$$\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right\}\mathrm{X}=-\frac{du}{dy}\left(y\,\frac{du}{dx}-x\,\frac{du}{dy}\right), \qquad \ldots\ldots\ldots\ldots (145)$$

$$\left\{\left(\frac{du}{dx}\right)^2+\left(\frac{du}{dy}\right)^2\right\}\mathrm{Y}=\ \frac{du}{dx}\left(y\,\frac{du}{dx}-x\,\frac{du}{dy}\right), \qquad \ldots\ldots\ldots\ldots (146)$$

whence it follows that if $(\xi,\ \eta)$ be the point inverse to (X, Y), the coordinates are given by

$$\xi=\frac{k^2\,\mathrm{X}}{\mathrm{X}^2+\mathrm{Y}^2}=-k^2.\frac{\dfrac{du}{dy}}{y\,\dfrac{du}{dx}-x\,\dfrac{du}{dy}} \qquad \ldots\ldots\ldots\ldots (147)$$

$$\eta=\frac{k^2\,\mathrm{Y}}{\mathrm{X}^2+\mathrm{Y}^2}=\ k^2.\frac{\dfrac{du}{dx}}{y\,\dfrac{du}{dx}-x\,\dfrac{du}{dy}} \qquad \ldots\ldots\ldots\ldots (148)$$

Therefore, the equation of the reciprocal polar of the evolute of the curve given by (142) is obtained by eliminating x and y from the three equations (142), (147), (148); and, the general theory being thus given, the question is reduced to one of elimination.

It is interesting to note that if the coordinates of any point on the given curve can be expressed in terms of a single variable parameter ϕ, the coordinates of the corresponding point on the reciprocal polar of the evolute, may be similarly expressed. For, remembering that

$$\frac{\frac{du}{dx}}{\frac{du}{dy}} = -\frac{dy}{dx},$$

the formulæ in (147) and (148) may be written

$$\xi = k^2 . \frac{1}{y\frac{dy}{dx}+x} = k^2 . \frac{\frac{dx}{d\phi}}{y\frac{dy}{d\phi}+x\frac{dx}{d\phi}}$$

$$\eta = k^2 . \frac{\frac{dy}{dx}}{y\frac{dy}{dx}+x} = k^2 . \frac{\frac{dy}{d\phi}}{y\frac{dy}{d\phi}+x\frac{dx}{d\phi}},$$

so that, if the coordinates of any point on the given curve be given by

$$x = f_1(\phi)$$
$$y = f_2(\phi),$$

we see at once that the coordinates of the corresponding point on the reciprocal polar of the evolute are given by the system

$$\xi = k^2 . \frac{f'_1(\phi)}{f_1(\phi) f'_1(\phi) + f_2(\phi) f'_2(\phi)}.$$

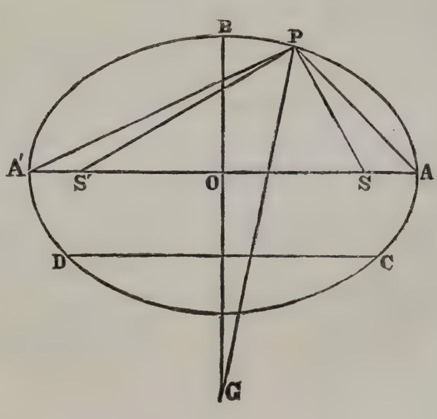
$$\eta = k^2 . \frac{f'_2(\phi)}{f_1(\phi) f'_1(\phi) + f_2(\phi) f'_2(\phi)}.$$

It is clear that the coordinates of any point on the n^{th} "reciprocal polar of evolute" may be obtained from this system; and the coordinates of points on the curves given above may also be expressed by means of a single variable parameter.

§§. 28—29. *Theorems on Central Conics.*

§. 28. **Properties of the Ellipse.**—In this section we shall investigate the truth of some theorems on the ellipse.

I. Let ϕ be the eccentric angle at any point P on the ellipse



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

so that, if A, A′ are the vertices and S, S′ the foci, the coordinates of A, A′, S, S′, P are (a, o), $(-a, o)$, (ae, o), $(-ae, o)$, $(a \cos\phi, b \sin\phi)$, respectively. The equations to PA, PS, PS′, PA′ are easily found, *viz.*,

$$\text{PA is} \quad \frac{x-a\cos\phi}{y-b\sin\phi}=\frac{a\cos\phi-a,}{b\sin\phi},$$

$$\text{or} \quad y=\frac{b}{a}\frac{\sin\phi}{\cos\phi-1}x-\frac{b\sin\phi}{\cos\phi-1} \qquad \text{............ (149)}$$

$$\text{PS is} \quad \frac{x-a\cos\phi}{y-b\sin\phi}=\frac{a\cos\phi-ae}{b\sin\phi},$$

$$\text{or} \quad y=\frac{b}{a}\frac{\sin\phi}{\cos\phi-e}x-\frac{be\sin\phi}{\cos\phi-e} \qquad \text{............ (150)}$$

$$\text{PS}' \text{ is} \quad \frac{x-a\cos\phi}{y-b\sin\phi}=\frac{a\cos\phi+ae}{b\sin\phi},$$

$$\text{or} \quad y=\frac{b}{a}\frac{\sin\phi}{\cos\phi+e}x+\frac{be\sin\phi}{\cos\phi+e} \qquad \text{............ (151)}$$

$$\text{PA}' \text{ is} \quad \frac{x-a\cos\phi}{y-b\sin\phi}=\frac{a\cos\phi+a}{b\sin\phi},$$

$$\text{or} \quad y=\frac{b}{a}\frac{\sin\phi}{\cos\phi+1}x+\frac{b\sin\phi}{\cos\phi+1} \qquad \text{............ (152)}$$

Let p, q be the intercepts made by PA, PA′, and r, s those made by PS, PS′, on the minor axis. Then we have

$$p=\frac{b\sin\phi}{1-\cos\phi}, \quad q=\frac{b\sin\phi}{1+\cos\phi},$$

$$r=\frac{be\sin\phi}{e-\cos\phi}, \quad s=\frac{be\sin\phi}{e+\cos\phi};$$

so that we get

$$p+q=\frac{2b}{\sin\phi}, \quad pq=b^2, \quad \frac{1}{p}+\frac{1}{q}=\frac{2}{b\sin\phi}.$$

$$r+s=\frac{2be^2\sin\phi}{e^2-\cos^2\phi}, \quad rs=\frac{b^2e^2\sin^2\phi}{e^2-\cos^2\phi}, \quad \frac{1}{r}+\frac{1}{s}=\frac{2}{b\sin\phi}.$$

This shews that the sum of the reciprocals of the intercepts made by PA, PA′ on the minor axis is equal to the sum of the reciprocals of the intercepts made by PS, PS′ on the same axis; it also follows that, since $pq=b^2$, the rectangle under the intercepts made by PA, PA′ is always constant and equal to the square of the semi-axis-minor. Again, p, q are the roots of the quadratic

$$z^2-2b\operatorname{cosec}\phi.\, z+b^2=0. \qquad \text{............ (153)}$$

Similarly, r, s are the roots of the quadratic

$$z^2-2b\,\lambda^2\operatorname{cosec}\phi.\, z+b^2\lambda^2=0 \qquad \text{............ (154)}$$

where λ^2 satisfies the equation

$$\lambda^2=\frac{e^2\sin^2\phi}{e^2-\cos^2\phi},$$

which is equivalent to

$$\frac{\sin^2\phi}{e^2-1}=\frac{\lambda^2}{e^2-\lambda^2}.$$

Again, since the equations of all the four lines PA, PA′, PS, PS′, are known, the angle between any two of them may be found, *viz.*,

$$\tan \text{APA}'=\frac{2\,ab}{(a^2-b^2)\sin\phi} \quad \text{.........} \quad (155)$$

$$\tan \text{SPS}'=\frac{2be}{a}.\frac{\sin\phi}{1-e^2\,(1+\sin^2\phi)} \quad \text{.........} \quad (156)$$

$$\cot \text{SPA}=-\frac{a}{b}.\frac{1+e}{1-e}\left\{\tan\frac{\phi}{2}-\frac{e^2}{1+e}\sin\phi\right\} \quad \text{.........} \quad (157)$$

$$\cot \text{S}'\text{PA}'=\frac{a}{b}.\frac{1+e}{1-e}\left\{\cot\frac{\phi}{2}-\frac{e^2}{1+e}\sin\phi\right\}. \quad \text{.........} \quad (158)$$

We have shewn above that

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{1}{s}=\frac{2}{b\sin\phi}=\frac{2}{\text{ordinate of P}},$$

whence the ordinate of P is a harmonic mean as well between r and s as between p and q. Again, it is evident that the theorem holds, even if S, S′ are not the foci, but any two points on the major axis equidistant from the centre; for, in that case, instead of putting $\text{OS}=ae$, we have to put $\text{OS}=ak$, where k is a certain constant; thus, we have the theorem that the ordinate of any point P is a harmonic mean between the intercepts made on the minor axis by the two lines joining P to two points on the major axis equidistant from the centre.

In order to see whether the formulæ

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{1}{s}=\frac{2}{y},$$

$$pq=k^2,$$

hold for any curve other than the conic, let us take the inverse question in a more general form, *viz.*, take O as the origin of coordinates, and BOA, OQP any two lines through it, A, B being fixed points; then, if BQ and AP intersect in R, required the locus of R, when

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+\frac{1}{s}=\frac{2}{y},$$

$$pq=k^2,$$

where $\text{OP}=p$, $\text{OQ}=q$. Let α, β be the coordinates of R; $\text{OA}=a$, $\text{OB}=-b$; then

$$\text{RA is }\frac{x-a}{y-\beta}=\frac{\alpha-a}{\beta},$$

$$\text{RB is }\frac{x-a}{y-\beta}=\frac{a+b}{\beta}.$$

But, since OP $= p$, OQ $= q$, we have

$$\frac{-a}{p-\beta} = \frac{\alpha - a}{\beta}, \quad \frac{-a}{q-\beta} = \frac{\alpha + b}{\beta},$$

whence

$$p = \frac{a\beta}{a - \alpha}, \quad q = \frac{b\beta}{\alpha + b}, \qquad \text{.......... (159), (160)}$$

so that

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{\beta} + \frac{\alpha\,(a-b)}{ab\,\beta}.$$

Hence the theorem that the ordinate is a harmonic mean between the intercepts holds only when $a = b$, that is, when the line on which the intercepts are made is equidistant from the fixed points; thus, we have the

Theorem.—Given two points and a line equidistant from them; then, taking for axes the given line and the line joining the points, the ordinate of any point is a harmonic mean between the intercepts which the lines joining the point to the given points make on the given line.

Again, if $pq = k^2$, we must have, changing α, β into x, y in (159) and (160),

$$\frac{ay}{a-x} \cdot \frac{by}{x+b} = k^2,$$

which may be written

$$\frac{x^2}{ab} + \frac{y^2}{k^2} + \left(\frac{1}{a} - \frac{1}{b}\right) x = 1,$$

shewing that the theorem holds only when P lies on a conic. In the particular case when the given line is equidistant from the given points, we have $a = b$, and the conic is

$$\frac{x^2}{a^2} + \frac{y^2}{k^2} = 1.$$

If the two lines are also at right angles, they are the axes of the conic, and the given constant k is the semi-axis-minor.

II. To determine the position of a point P on an ellipse such that, if the normal at P intersects the minor axis produced in G, the polar of G may subtend a right angle at P.

Using the same diagram, let the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and P the required point where the eccentric angle is ϕ, so that the coordinates of P are $a \cos \phi$, $b \sin \phi$. Then the normal at P is

$$\frac{ax}{\cos \phi} - \frac{by}{\sin \phi} = c^2,$$

so that G is

$$\left(0, -\frac{c^2 \sin \phi}{b}\right).$$

Let CD be the polar of G with respect to the conic, so that CD is parallel to the axis-major and has for its equation

$$y = -\frac{b^3}{c^2 \sin\phi}.$$

Transfer the origin to P, and take the new axes parallel to the old; then the ellipse is

$$\frac{(x+a\cos\phi)^2}{a^2} + \frac{(y+b\sin\phi)^2}{b^2} = 1$$

or $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2\cos\phi}{a}x + \frac{2\sin\phi}{b}y = 0 \quad \text{........... (161)}$$

The line CD is

$$y + b\sin\phi = -\frac{b^3}{c^2\sin\phi},$$

or $$y = \lambda \quad \text{........... (162)}$$

where $$\lambda = -\frac{b}{c^2\sin\phi}(a^2\sin^2\phi + b^2\cos^2\phi) \quad \text{........... (163)}$$

Now, PD, PC are two lines through the new origin, and through the intersection of the conic with the line; their equation, therefore, must be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2\cos\phi}{\lambda a}xy + \frac{2\sin\phi}{\lambda b}y^2 = 0 \quad \text{........... (164)}$$

These will be at right angles, if

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{2\sin\phi}{\lambda b} = 0.$$

Substituting for λ from (163) and simplifying, we have

$$\sin^2\phi = \left(1 - \frac{1}{e^2}\right)\left(1 - \frac{2}{e^2}\right) \quad \text{........... (165)}$$

which determines the value of ϕ, and, therefore, of P; it is remarkable that the result is dependent simply on the eccentricity.

III. A very interesting point arises, if we seek the envelope of the sides of any triangle PSS′ having its vertex P at any point on the ellipse, and its base-ends any two points S, S′ on the axis-major, equidistant from the centre, so that OS = OS′ = k. Then, from (150), the equation of PS is

$$y = \frac{b}{a}\cdot\frac{\sin\phi}{\cos\phi - k}x - \frac{bk\sin\phi}{\cos\phi - k},$$

which may be written

$$(bx - akb)\sin\phi - ay\cos\phi = -aky,$$

and the envelope of this for different values of ϕ is

$$(bx - akb)^2 + a^2y^2 = a^2k^2y^2, \quad \text{........... (166)}$$

which is equivalent to

$$b^2(x - ak)^2 = a^2(k^2 - 1)y^2 \quad \text{........... (167)}$$

or $$b(x - ak) = \pm a\sqrt{k^2 - 1}\,y;$$

apparently, therefore, the envelope is a pair of right lines passing through the fixed point (ak, o), and real only if k is greater than unity, that is, if the point S is outside the ellipse. But, looking to the geometry of the figure, it is clear that the envelope must be the given point S, so that the analytical solution furnishes, apparently, a whole line for the envelope, while geometrically only one definite point on that line satisfies the demand of the problem; the discrepancy, however, is only apparent, *viz.*, the equation (167) may be written

$$b^2(x-ak)^2+a^2\left(\sqrt{1-k^2}\right)^2 y^2=0,$$

so that this must be equivalent to

$$\left.\begin{array}{l} x=ak \\ y=0 \end{array}\right\},$$

which is, of course, the point in question. Such instances of degenerate envelopes are by no means rare.

§. 29. **Properties of Confocals.**

I. Given a system of confocal ellipses, to find the locus of points where the tangents cut off a constant area from the axes.

Any conic of the system is

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1, \qquad \text{............ (168)}$$

where, for the moment,

$$A^2=a^2+\lambda^2,\ B^2=b^2+\lambda^2,\ c^2=A^2-B^2=a^2-b^2.$$

Take a point (ξ, η) on this ellipse where the eccentric angle is ϕ; the tangent is

$$\frac{x}{A}\cos\phi+\frac{y}{B}\sin\phi=1,$$

and the intercepts made on the axes are

$$\frac{A}{\cos\phi},\ \frac{B}{\sin\phi},$$

so that, if h^2 be double the constant area in question, we have

$$\frac{AB}{\sin\phi\cos\phi}=h^2 \qquad \text{............ (169)}$$

Hence we get the system

$$\xi^2=A^2\cos^2\phi=(a^2+\lambda^2)\cos^2\phi, \qquad \text{............ (170)}$$

$$\eta^2=B^2\sin^2\phi=(b^2+\lambda^2)\sin^2\phi, \qquad \text{............ (171)}$$

and from (169)

$$(a^2+\lambda^2)(b^2+\lambda^2)=h^4\sin^2\phi\cos^2\phi. \qquad \text{............ (172)}$$

The elimination of λ, ϕ from these three equations will lead us to the equation of the locus. For this purpose, observe that from (170) and (171),

$$\xi^2\eta^2=(a^2+\lambda^2)(b^2+\lambda^2)\sin^2\phi\cos^2\phi=h^4\sin^4\phi\cos^4\phi,$$

so that

$$\xi\eta = h^2 \sin^2\phi \cos^2\phi. \qquad \text{............} \; (173)$$

Again, from (170) and (171),

$$\frac{\xi^2}{\cos^2\phi} - \frac{\eta^2}{\sin^2\phi} = a^2 - b^2 = c^2,$$

or

$$\xi^2 \sin^2\phi - \eta^2 \cos^2\phi = c^2 \sin^2\phi \cos^2\phi = \frac{c^2\,\xi\eta}{h^2},$$

from (173).

This may be written

$$\xi^2 \sin^2\phi - \eta^2 (1 - \sin^2\phi) = \frac{c^2}{h^2}\,\xi\eta,$$

whence

$$\sin^2\phi = \frac{\eta^2}{\xi^2+\eta^2} + \frac{c^2}{h^2}\,\frac{\xi\eta}{\xi^2+\eta^2}, \qquad \text{............} \; (174)$$

$$\cos^2\phi = \frac{\xi^2}{\xi^2+\eta^2} - \frac{c^2}{h^2}\,\frac{\xi\eta}{\xi^2+\eta^2}. \qquad \text{............} \; (175)$$

Substituting for $\sin\phi$ and $\cos\phi$ from (174) and (175) in (173), and simplifying, we have

$$(c^2\xi + h^2\eta)(h^2\xi - c^2\eta) = h^2\,(\xi^2+\eta^2)^2,$$

which is the equation of the locus in question. Hence, we have the theorem that the locus of points on a system of confocal ellipses where the tangents cut off a constant area from the axes is the bicircular quartic through the origin

$$(c^2x + h^2y)(h^2x - c^2y) = h^2\,(x^2+y^2)^2, \qquad \text{............} \; (176)$$

where c is half the distance between the foci, and h^2 double the given constant area.

It is not difficult to see that this quartic-locus is the inverse of a central conic, for, substituting for x and y

$$\frac{k^2x}{x^2+y^2}, \quad \text{and} \quad \frac{k^2y}{x^2+y^2}$$

respectively, we find that the bicircular quartic is the inverse of the conic

$$(c^2x + h^2y)(h^2x - c^2y) = h^2 k^4, \qquad \text{............} \; (177)$$

where k is the radius of inversion; it is easy to see that this conic is an equilateral hyperbola concentric with the confocal ellipses, and, if θ be the inclination of its transverse axis to the line joining the foci of the confocal family, we have

$$\tan 2\theta = \tfrac{1}{2}\left(\frac{h^2}{c^2} - \frac{c^2}{h^2}\right),$$

which furnishes for $\tan\theta$ the two values

$$\frac{h^2 - c^2}{h^2 + c^2}, \quad \frac{c^2 + h^2}{c^2 - h^2}.$$

II. To investigate the locus of points on a system of confocal ellipses, where the eccentric angle has a constant value.

Let any one of the confocal system be

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$$

where $A^2=a^2+\lambda^2$, $B^2=b^2+\lambda^2$; then, if ϕ be the eccentric angle at any point (ξ, η), we have

$$\xi^2=A^2\cos^2\phi=(a^2+\lambda^2)\cos^2\phi,$$
$$\eta^2=B^2\sin^2\phi=(b^2+\lambda^2)\sin^2\phi,$$

so that the locus in question is the hyperbola

$$\frac{\xi^2}{\cos^2\phi}-\frac{\eta^2}{\sin^2\phi}=a^2-b^2=c^2, \qquad \ldots\ldots\ldots\ldots (178)$$

and this is evidently a member of the confocal family; hence it follows that, given a system of confocal ellipses, the locus of points where the eccentric angle has a constant value is one of the confocal hyperbolas which intersect the system orthogonally; in other words, given a confocal system of ellipses and hyperbolas, each hyperbola intersects the ellipses at points where the eccentric angle has a constant value, and, by variation of this constant value, we get all the hyperbolas of the system, and, from a known theorem, the envelope of all these hyperbolas is an imaginary quadrilateral.

Similarly, if we have the hyperbola

$$\frac{x^2}{a^2+\lambda^2}-\frac{y^2}{b^2+\lambda^2}=1,$$

which is one of a confocal system, and ϕ the eccentric angle at any point (ξ, η), we have

$$\xi^2=(a^2+\lambda^2)\sec^2\phi,$$
$$\eta^2=(b^2+\lambda^2)\tan^2\phi,$$

so that, if the eccentric angle has a constant value, the locus is

$$\frac{\xi^2}{\sec^2\phi}-\frac{\eta^2}{\tan^2\phi}=a^2-b^2=c^2 \qquad \ldots\ldots\ldots\ldots (179)$$

and the envelope of this, for different values of the eccentric angle, is the parallelogram formed by the four lines

$$(c^2+y^2-x^2)^2=4c^2y^2, \qquad \ldots\ldots\ldots\ldots (180)$$

viz., the four lines are

$$-c+y+x=0,\ c-y+x=0,\ c+y-x=0,\ c+y+x=0.$$

§§. 30—31. *Theorems on the Parabola.*

§. 30. **A Dynamical Problem.**—Take the parabola

$$y^2=4ax,$$

which, when the origin is removed to a point on the principal axis at a distance na from the vertex, becomes

$$y^2=4a\,(x+na). \qquad \ldots\ldots\ldots\ldots (181)$$

Imagine a particle to describe the parabola under the action of a force directed to the new origin as centre; and suppose it to be started from the apse with the velocity in a circle at the same distance. Then

$$2y\frac{dy}{dt}=4a\frac{dx}{dt},$$

and
$$\left(\frac{dy}{dt}\right)^2+y\frac{d^2y}{dt^2}=2a\frac{d^2x}{dt^2}.$$

But
$$x\frac{dy}{dt}-y\frac{dx}{dt}=h,$$

so that
$$x\frac{dy}{dt}-\frac{y^2}{2a}\frac{dy}{dt}=h,$$

whence
$$(x+2na)\frac{dy}{dt}=-h.$$

Therefore
$$\frac{h^2}{(x+2na)^2}-\mathrm{P}\frac{y^2}{r}=-2a.\,\mathrm{P}.\,\frac{x}{r},$$

where P is the central force.

This may be written

$$\frac{h^2}{(x+2na)^2}=\frac{\mathrm{P}}{r}(y^2-2ax)=\frac{\mathrm{P}}{r}.\,2a\,(x+2na),$$

which gives

$$\mathrm{P}=\frac{h^2}{2a}\frac{r}{(x+2na)^3}. \qquad \text{.......... (182)}$$

But
$$x^2+y^2=r^2$$
$$y^2=4a\,(x+na).$$

Eliminating y, this gives a quadratic for x, whence we derive

$$x+2na=2a\,(n-1)+\left\{r^2+4a^2(1-n)\right\}^{\frac{1}{2}}.$$

Substituting in (182), we get

$$\mathrm{P}=\frac{h^2}{2a}\frac{r}{\left\{2a\,(n-1)+\sqrt{r^2+4a^2\,(1-n)}\right\}^3} \qquad \text{.......... (183)}$$

which gives the law of force in terms of the radius vector. For an interesting discussion of a kinetic difficulty in connection with this dynamical problem, see a note by Dr. Besant in the *Quarterly Journal of Mathematics*, t. XI, 38.

§. 31. **Geometrical Applications.**—Thus far we have solved a purely dynamical question; we now proceed to obtain some interesting geometrical properties of the parabola. We have

$$\mathrm{P}=\frac{h^2}{p^3}\frac{dp}{dr}=-\frac{h^2}{2}\frac{d}{dr}\left(\frac{1}{p^2}\right).$$

Hence, from (183), we get

$$-\frac{a}{p^2}=\int\frac{rdr}{\left\{2a\ (n-1)+\sqrt{\left[r^2+4a^2\,(1-n)\right]}\right\}^3}.$$

If, therefore, we take p for all values of r from $+\infty$ to $-\infty$, we have

$$\Sigma\left(-\frac{a}{p^2}\right)=2\int_\infty^{na}\frac{rdr}{\left\{2a\ (n-1)+\sqrt{\left[r^2+4a^2\,(1-n)\right]}\right\}^3} \qquad (184)$$

To evaluate this definite integral, let us first take the indefinite form. Put

$$r^2=4a^2\ (1-n)\ \tan^2\phi, \qquad \ldots\ldots \quad \ldots\ (185)$$

$$r=2a\sqrt{1-n}\ \tan\phi,$$

$$dr=2a\sqrt{1-n}\ \sec^2\phi\ d\phi,$$

$$r^2+4a^2\ (1-n)=4a^2\ (1-n)\ \sec^2\phi.$$

If, therefore, I be the indefinite integral, we have

$$\text{I}=\int\frac{4a^2\ (1-n)\ \tan\phi\ \sec^2\phi\ d\phi}{\left\{2a\ (n-1)+2a\sqrt{1-n}\ \sec\phi\right\}^3}$$

$$=\int\frac{4a^2\ (1-n)\ \sin\phi\ d\phi}{\left\{2a\sqrt{1-n}-2a\ (1-n)\ \cos\phi\right\}^3}$$

$$=\int\frac{4a^2\ (1-n)\ \sin\phi\ d\phi}{8a^3\ (1-n)^{\frac{3}{2}}\left\{1-\sqrt{1-n}\ \cos\phi\right\}^3}$$

$$=-\frac{1}{2a\sqrt{1-n}}\int\frac{d\ (\cos\phi)}{\left\{1-\sqrt{1-n}\ \cos\phi\right\}^3}$$

$$=\frac{1}{4a\ (n-1)}\ \frac{1}{\left\{1-\sqrt{1-n}\ \cos\phi\right\}^2}. \qquad \ldots\ldots\ (186)$$

Now, $\sec^2\phi=1+\tan^2\phi=1+\dfrac{r^2}{4a^2\ (1-n)}$, from (185).

Therefore

$$\cos^2\phi=\frac{4a^2\ (1-n)}{r^2+4a^2\ (1-n)},$$

and, when $r=na$, this gives

$$\cos^2\phi=\frac{4\ (1-n)}{(2-n)^2}$$

and, when $r=\infty$,

$$\cos^2\phi=0.$$

These give the limits of the transformed integral; if, therefore, Q be the value of the definite integral, we have

$$Q = -\frac{1}{an^2},$$

so that, from (184), we have

$$a \Sigma \left(-\frac{1}{p^2}\right) = 2Q = -\frac{2}{an^2},$$

whence, finally,

$$\Sigma \left(\frac{1}{p^2}\right) = \frac{2}{a^2n^2}. \qquad \text{............ (187)}$$

Hence we have the theorem that, if we take any point on the axis of a parabola whose distance from the vertex is na, the sum of the squares of the reciprocals of all the perpendiculars dropped from this point on successive tangents to the parabola is equal to $\frac{2}{n^2a^2}$. It is obvious that these perpendiculars are the radii-vectores of a pedal of the parabola; hence, the following theorems may be enunciated.

Theorem I.—A is the vertex and S_1 the focus of a parabola whose latus-rectum is $4a$; points $S_2, S_3, \ldots\ldots S_\infty$ are taken on the principal axis such that $AS_1 = S_1 S_2 = \ldots = a$; the sum of the squares of the reciprocals of the radii-vectores of the pedal of the parabola with regard to S_n is $\frac{2}{n^2a^2}$. (188)

Theorem II.—The sum of the squares of the reciprocals of the radii-vectores of all the pedals of the parabola with regard to $S_1, S_2 \ldots S_\infty$ is

$$= \frac{2}{a^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \ldots\ldots\right) = \frac{1}{3}\left(\frac{\pi}{a}\right)^2 \qquad \text{......... (189)}$$

Theorem III.—If we take only the odd pedals, the sum of the squares of the reciprocals of all the radii-vectores is

$$= \frac{2}{a}\left(\frac{1}{1^2} + \frac{1}{3^2} + \ldots\ldots\right) = \frac{1}{4}\left(\frac{\pi}{a}\right)^2 \qquad \text{......... (190)}$$

Theorem IV.—If we take only the even pedals, the sum of the squares of the reciprocals of all the radii-vectores is

$$= \frac{2}{a}\left(\frac{1}{2^2} + \frac{1}{4^2} + \ldots\ldots\right) = \frac{1}{12}\left(\frac{\pi}{a}\right)^2 \qquad \text{......... (191)}$$

§. 32. *A Geometrical Locus.*

§. 32. **General Theorem on Conics.**—If from any point P two tangents be drawn to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{.............. (192)}$$

to investigate the locus of the middle point of the chord of contact when

P is constrained to move on any curve

$$F\ (x, y) = 0. \qquad \ldots\ldots\ldots\ldots\ldots \quad (193)$$

Let θ, ϕ be the eccentric angles at the points of contact of the tangents; then the tangents are

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1,$$

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1,$$

and, if X, Y be the coordinates of P, we have

$$X = a. \frac{\cos \frac{\theta+\phi}{2}}{\cos \frac{\theta-\phi}{2}}.$$

$$Y = b. \frac{\sin \frac{\theta+\phi}{2}}{\cos \frac{\theta-\phi}{2}}.$$

If, further, ξ, η be the coordinates of the middle point of the chord of contact the locus of which is sought, we have

$$\xi = \frac{a}{2} (\cos \theta + \cos \phi) \qquad \ldots\ldots\ldots\ldots \quad (194)$$

$$\eta = \frac{b}{2} (\sin \theta + \sin \phi) \qquad \ldots\ldots\ldots\ldots \quad (195)$$

The locus is obtained by eliminating θ, ϕ between these and

$$F \left\{ a \frac{\cos \frac{\theta+\phi}{2}}{\cos \frac{\theta-\phi}{2}},\ b \frac{\sin \frac{\theta+\phi}{2}}{\cos \frac{\theta-\phi}{2}} \right\} = 0 \quad \ldots \quad (196)$$

From (194) and (195), we have

$$\frac{\xi}{a} = \cos \frac{\theta+\phi}{2} \cos \frac{\theta-\phi}{2}$$

$$\frac{\eta}{b} = \sin \frac{\theta+\phi}{2} \cos \frac{\theta-\phi}{2}.$$

whence, squaring and adding,

$$\cos^2 \frac{\theta-\phi}{2} = \frac{\xi^2}{a^2} + \frac{\eta^2}{b^2}. \qquad \ldots\ldots\ldots\ldots \quad (197)$$

Also, by division, from (194) and (195),

$$\tan \frac{\theta+\phi}{2} = \frac{a\eta}{b\xi}$$

whence

$$\sin \frac{\theta+\phi}{2} = \frac{a\eta}{\sqrt{b^2\xi^2 + a^2\eta^2}},\quad \cos \frac{\theta+\phi}{2} = \frac{b\xi}{\sqrt{b^2\xi^2 + a^2\eta^2}} \quad \ldots\ldots\ldots \quad (198),\ (199)$$

Substituting from (197), (198), and (199) in (196), the equation of the locus sought is found to be

$$F\left(\frac{a^2b^2\xi}{b^2\xi^2+a^2\eta^2},\ \frac{a^2b^2\eta}{b^2\xi^2+a^2\eta^2}\right)=0. \quad \text{..........} \quad (200)$$

We have, therefore, the

Theorem.—If from any point P, tangents are drawn to the conic

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

and P is constrained to move on any curve

$$F\ (x,y)=0,$$

the locus of the middle point of the polar chord of P with regard to S is

$$F\left(\frac{x}{1+S},\ \frac{y}{1+S}\right)=0.$$

Similarly, if we consider the parabola

$$y^2=4ax,$$

any two points on the curve are

$$(a\tan^2\theta,\ 2a\tan\theta),\ (a\tan^2\phi,\ 2a\tan\phi),$$

so that the coordinates of the point of intersection of the tangents are given by

$$X=a\tan\theta\tan\phi$$
$$Y=a\ (\tan\theta+\tan\phi),$$

and the middle point of the polar chord is given by

$$\xi=\frac{a}{2}\ (\tan^2\theta+\tan^2\phi),$$

$$\eta=a\ (\tan\theta+\tan\phi).$$

These give

$$\frac{\eta^2}{a^2}=\frac{2\xi}{a}+2\tan\theta\tan\phi,$$

whence

$$X=\frac{\eta^2-2a\ \xi}{2a},\ Y=\eta.$$

Hence, substituting in $F\ (x,y)=0$, we have the

Theorem.—If from any point P tangents are drawn to the parabola

$$y^2=4ax,$$

and P is constrained to move on the curve

$$F\ (x,y)=0,$$

the locus of the middle point of the polar chord of P with regard to the parabola is

$$F\left(\frac{y^2-2ax}{2a},\ y\right)=0.$$

We will here simply add that the result obtained above in equation (200) is an immediate consequence of a new method which we propose to call the **Method of Elliptic Inversion.**

26*th October*, 1887.